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Abstract
The performances of various optical elements are presented for ideally monochromatic X-rays in terms of position–angle space, which is used in the phase-space method. The optical elements discussed are a synchrotron radiation source, flat perfect crystals (reflection and transmission geometries), mosaic crystals, curved crystals of reflection (Johansson, Johann and logarithmic spiral types) and transmission geometries, and an elliptic total reflection mirror. The angular widths of acceptance and emergence for diffraction are properly taken into account in both the flat and curved crystals. It is shown that focusing optical elements such as a curved crystal and an elliptical mirror should not be treated simply as lenses.

1. Introduction
Optimumlly designed X-ray optical systems are indispensable in making the best use of a synchrotron radiation source which has drastically different characteristics from conventional X-ray sources. Some suitable methods to describe optical properties of various optical elements and their combined systems are required for the estimation of the performance of a given X-ray optical system. This requirement has partially been met by the DuMond (1937) diagram and the phase-space method (Green, 1976; Pianetta & Lindau, 1976, 1977; Hastings, 1977). In the DuMond diagram the position of an X-ray photon is not described, while in the phase-space method the wavelength-dependent properties are not explicitly described. In this sense the methods are complimentary.

In previous papers (Matsushita, 1978; Matsushita, Kaminaga & Kohra, 1978) brief and preliminary reports were made with emphasis on the following two points: (1) angular widths of acceptance and emergence for diffraction should be taken into account in the position–angle space for both flat and curved crystals; and (2) concepts used in the method of the DuMond diagram and in the phase-space method can be combined by introducing a position–angle–wavelength space.

In the present and following (part II: Matsushita & Kaminaga, 1980) papers, we will give a systematic method of estimating performances of X-ray optical systems. In the present paper, our discussion is limited to the behavior of various X-ray optical elements for ideally monochromatic X-rays of wavelength \( \lambda_0 \). Basic representations of various optical elements are given in the position–angle space. These will be the basis for our discussion of optical systems for polychromatic X-rays using the position–angle–wavelength space in part II.

2. Synchrotron radiation source
Although synchrotron radiation is polychromatic, only one wavelength component, \( \lambda_0 \), is considered in the present paper. The plane of electron orbit of a synchrotron or storage ring is taken to be horizontal. The x and y axes of the Cartesian coordinates are taken in the horizontal and vertical planes, respectively, and the z axis, which can be considered as the optical axis, is taken along a tangent to the electron orbit. The angles between the X-ray beam direction and the z axis are denoted by \( x' \) and \( y' \) in the horizontal (x–z) and vertical (y–z) planes, respectively.

The description of a synchrotron radiation source in position–angle space has been discussed in detail by Green (1976). A representation is briefly summarized...
here in order to acquire useful expressions for ensuing discussions in part II.

The electron phase ellipse is represented by

\[ y' = -\frac{\alpha}{\beta} y + \frac{1}{\beta} (\beta E - y^2)^{1/2} \]  

(1)

in the \( y-y' \) space, where \( \alpha \) and \( \beta \) are parameters which characterize the electron phase ellipse shown by \( EPE \) in Fig. 1(a) and \( E \) is the emittance of the electron beam (Green, 1976). The overall distribution of emitted photons is obtained by carrying out a convolution integral between the angular distribution function of photons emitted by a relativistic electron and that of electrons in the orbit over the angle. If the angular distribution of emitted photons from a relativistic electron is approximated by a Gaussian function, the standard deviation of which is denoted by \( \sigma \), the one sigma contour of the resulting distribution is given by

\[ y' = \frac{cz}{\sqrt{1 + \frac{\gamma^2}{c^2}}} \left[ \frac{y + \alpha}{\beta} \right] \]  

(2)

and is represented by \( SY \) in Fig. 1(a). Notice that the two ellipses, \( EPE \) and \( SY \), have a common diameter \( DD' \), the slope of which is given by \(-\alpha/\beta\). This means that the central direction of emitted photons at a point displaced by \( y \) should coincide with that of the electron beam at \( y \): \( PM = P'M \) and \( QM = Q'M \).

Pianetta & Lindau (1976, 1977) and Hastings (1977) showed an ellipse, whose diameter has steeper slopes than that of the electron phase ellipse, as an overall photon distribution. According to their expression, the central direction of the X-ray beam emitted at a displacement \( y \) from the optical axis does not coincide with that of the electron beam at \( y \). However, the directions should coincide since each electron emits X-ray photons symmetrical in angle with respect to the direction of its motion.

The X-ray beam which has traveled through a distance \( z \) along the optical axis in the vacuum is represented by \( BY \) in Fig. 1(b). If a slit of width \( 2y_s \) is placed in the beam path there, an X-ray beam represented by \( U \) in Fig. 1(b) is transmitted through the slit. Quantities such as the area \( S_{SY} \) of \( SY \), area \( S_U \) of \( U \), and angular divergences \( \Delta y'_m \) and \( \Delta y'_s \) are given by

\[ S_{SY} = \pi \left[ E \left( E + \beta \sigma^2 \right) \right]^{1/2}, \]  

(3)

\[ S_U = \frac{2 \left[ E \left( E + \beta \sigma^2 \right) \right]^{1/2}}{W^2} \left[ y_s \left( W^2 - y_s^2 \right)^{1/2} + W^2 \sin \frac{y_s}{W} \right], \]  

(4)

\[ \Delta y'_m = 2 \left[ E \left( \frac{y + \sigma^2}{E} \right) \right]^{1/2}, \]  

(5)

and

\[ \Delta y'_s = 2 \left[ \frac{E \left( y + \sigma^2 \right)}{E} \right]^{1/2} \left( \frac{y_s \left( W^2 - y_s^2 \right)^{1/2} + 2W^2 \sin \frac{y_s}{W}}{\left( y + \sigma^2 \right)^2 + 2 \sigma^2} \right) \]  

(6)

where

\[ W = \left\{ \left[ \left( y + \sigma^2 \right)^2 + 2 \sigma^2 \right] \right\}^{1/2}. \]  

Here, \( \gamma \) is a parameter given by \( \gamma = (1 + \alpha^2)/\beta \) (Green, 1976). Note that the area, \( S_{BY} \), covered by \( BY \) in Fig. 1(b) is always equal to \( S_{SY} \).

The synchrotron radiation source in the horizontal plane is represented by a parabolic band in the \( x-x' \) space (Green, 1976; Pianetta & Lindau, 1976, 1977; Hastings, 1977).

3. Flat crystals

3.1 Perfect crystal

(1) Reflection geometry. Fig. 2(a) shows the geometry of reflection for a perfect crystal using an asymmetric diffraction (Renninger, 1961; Kohra, 1962) where the diffracting lattice plane makes an angle \( \theta_p \) to the surface. The reflectivity of the crystal is almost equal to unity if the diffraction condition is satisfied. The angular widths, \( \omega_0 \) of acceptance and \( \omega_b \) of emergence, for diffraction are approximately given by

\[ \omega_0 = \frac{2}{\sin 2\theta_p} \frac{e^{2\lambda_0^2}}{\pi n v c^2} |F'_{hkl}| \left\{ \frac{1}{|b|^{1/2}} P \right\}, \]  

(8)

\[ \omega_b = \frac{2}{\sin 2\theta_p} \frac{e^{2\lambda_0^2}}{\pi n v c^2} |F'_{hkl}| \left\{ b \right\}^{1/2} P \]  

(9)

for a monochromatic X-ray beam of wavelength \( \lambda_0 \), where \( F'_{hkl} \) is the real part of the structure factor for the \( hkl \) reflection, \( \theta_p \) the geometrical Bragg angle, \( P \) the polarization factor given by \( P = 1 \) for \( \sigma \) polarization of X-rays or \( P = |\cos 2\theta_p| \) for \( \pi \) polarization, \( m \) and \( e \) are the mass and the charge of an electron, respectively, \( \nu \) is the volume of the unit cell of the crystal, \( c \) the velocity of light and \( b \) is given by

\[ b = \frac{\sin (\varphi + \theta_p)}{\sin (\varphi - \theta_p)}. \]  

(10)
\(\omega_0\) is smaller than \(\omega_h\) by \(1/|b|\) when \(|b| > 1\). At the same time the spatial width \(l_h\) of the emergent beam from the crystal is reduced to \(1/|b|\) of that, \(l_0\), of the accepted beam.

Two optical axes, \(SO\) and \(OT\), are taken in Fig. 2(a) parallel to central directions of the angular ranges of acceptance and emergence, for selective diffraction by the crystal for X-rays of wavelength \(\lambda_0\), respectively, so that \(OT\) makes an angle \(2\phi_0 + (2 + 1/|b| + |b|) \times |\psi_0|/2\sin 2\theta_0\) to \(SO\), where \(\psi_0\) is the real part of the 0th Fourier component of the polarizability of the crystal. The term \((2 + 1/|b| + |b|)|\psi_0|/2\sin 2\theta_0\) is due to the refraction effect and is of the order of seconds of arc. Off-axis displacements \(x_0\) and \(x_h\) are taken perpendicular to \(SO\) and \(OT\), respectively. Angles between the optical axes and the beam directions are denoted by \(\theta_0\) and \(\theta_h\), respectively. The senses of \(x_0\) and \(x_h\) are oppositely defined when referred to the signs of \(SO\) and \(OT\) optical axes. By taking \(OX_0\) and \(OX_h\) axes in the manner shown in Fig. 2(a) the positive values of \(x_0\) and \(x_h\) are taken parallel to \(SO\) and \(OT\), respectively. Hypothetical beam paths \(LP\) and \(QN\) are considered in place of the actual beam paths \(LM\) and \(MN\). When \(|x_0| < \omega_0/2\), the X-ray beam is accepted for diffraction at any point on the crystal surface and the point shown by \((x_0, x_0')\) in the position–angle space is transformed into \((x_h, x_h')\) following

\[
\begin{bmatrix}
  x_h \\
  x_h'
\end{bmatrix} = \begin{bmatrix}
  1/|b| & 0 \\
  0 & |b|
\end{bmatrix} \begin{bmatrix}
  x_0 \\
  x_0'
\end{bmatrix}.
\]

(11)

When \(|x_0| > \omega_0/2\), the X-ray beam cannot be accepted by the crystal. Hence, a flat perfect-crystal monochromator can be represented by windows in the \(x-x'\) space; the window \(A\) of acceptance is transformed into the window \(E\) of emergence as shown in Fig. 2(b).

Channel-cut crystals have fruitfully been used in X-ray optics for conventionally obtained X-rays (Bonse & Hart, 1965, 1966) and synchrotron radiation (Beaumont & Hart, 1974; Kincaid, 1975). As shown in Fig. 3, the surfaces of the walls of the groove need not necessarily be parallel to the diffracting lattice planes. For the case shown in Fig. 3, the transforming matrix \(T_c\) is given by

\[
T_c = \begin{bmatrix}
  |b| & 0 \\
  0 & 1/|b|
\end{bmatrix} \begin{bmatrix}
  1/|b| & 0 \\
  0 & |b|
\end{bmatrix} = \begin{bmatrix}
  1 & 0 \\
  0 & 1
\end{bmatrix}.
\]

(12)

This means that a channel-cut crystal with even-times reflections has a window of emergence which is identical to that of acceptance. If \(|b| > 1\), we can get a factor of \((|b|)^{1/2}\) larger angular width of the window than that of a symmetrically channel-cut crystal, as was pointed out previously (Kohra, Ando, Matsushita & Hashizume, 1978). By choosing \(\psi\) to make \(|b| < 1\), it is possible to have a much narrower angular width of the window.

(2) Transmission geometry. We will confine ourselves to the case of symmetric transmission, where the diffracting lattice plane is normal to the crystal surface; this is the most commonly used geometry and the treatment becomes much simpler. In this case the window of acceptance is identical to the window of emergence in the position–angle space. The full width at half maximum of the diffraction curve for the Laue case is taken as the angular width of the window. The transforming matrix from the window of acceptance to that of emergence is given by

\[
T_{is} = \begin{bmatrix}
  -1 & 0 \\
  0 & 1
\end{bmatrix}.
\]

(13)

This is in contrast with the reflection case using a symmetric diffraction, for which the transforming matrix is obtained by making \(|b| = 1\) in the matrix in (11).

3.2 Mosaic crystals

In a mosaic crystal the mosaic angular spread \(\Delta\) is

Fig. 2. A flat perfect crystal of the reflection type. (a) Geometry and definition of optical axes and off-axis displacements. The diffracting lattice plane makes an angle \(\phi\) to the surface. (b) The position–angle space representation. The window of acceptance, \(A\), is transformed into that of emergence, \(E\). See the text.

Fig. 3. The geometry of a channel-cut crystal monochromator.
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usually larger than a few or several minutes of arc, being much larger than the angular width of diffraction of each crystallite. We can consider that the beam reflected by a crystallite in the mosaic crystal is always deflected by an angle $2\theta_B$. As shown in Fig. 4(a), the beam making an angle $x_l = -\Delta \theta$ to the optical axis $SO$ is reflected in the direction making an angle $x_h = + \Delta \theta$ to the optical axis $OT$. As long as the absorption of X-rays is not extremely small, we can assume that X-ray beams are reflected in a region very close to the surface. Then the relation between $OP(-x_0)$ and $OQ(x_h)$ is given by

$$x_h = \frac{x_0}{|b|} \left[ 1 - \frac{\cot(\theta + \phi) + \cot(\theta - \phi)}{\cot(\theta + \phi) + \cot(\theta - \phi)} \right] \Delta \theta .$$

When $\Delta \theta$ is small and $\theta \pm \phi$ is not close to zero, the term $\frac{\cot(\theta + \phi) + \cot(\theta - \phi)}{\cot(\theta + \phi) + \cot(\theta - \phi)} \Delta \theta$ can be neglected. The mosaic-crystal monochromator is represented as windows in the position–angle space, where the point $(x_0, x'_0)$ is transformed into $(x_h, x'_h)$ as follows:

$$\begin{pmatrix} x_h \\ x'_h \end{pmatrix} = \begin{pmatrix} 1/|b| & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} .$$

In Fig. 4(b) the window $A(A_1A_2A_3A_4)$ of acceptance is transformed into the window $E(E_1E_2E_3E_4)$ of emergence. The angular widths of these windows are equal to the mosaic angular spread $\Delta$. The spatial width of the X-ray beam is reduced in the case shown in the figure. The area covered by $E$ is smaller than that covered by $A$, resulting in condensation of the spatial density of X-ray photons. This might seem to be a violation of Liouville's theorem. However, the situation is somewhat similar to the case of a beam scatterer in charged-particle beam optics (Banford, 1966); two photons incident at two different points on the same line parallel to the $Ox_0$ axis at the same time do not simultaneously reach points on the same line parallel to the $Ox_h$ axis.

When a divergent beam ($A_1A_3$ in Fig. 4b) emitted from a point source is incident on a mosaic-crystal monochromator, it is reflected as a convergent beam shown by $E_1E_3$. At a suitable distance from the crystal the X-ray beam can be represented by $F_1F_3$. This means that the X-ray beam is focused even if the crystal is not bent (Sparks, 1978).

4. Focusing optical elements

4.1 Curved crystals

A curved-crystal monochromator has a focusing property as well as the magnifying or reducing property. However, the monochromator should be treated differently from a lens because (1) at any point on the crystal surface only the X-ray beam incident in a narrow angular region is reflected, while the beam incident outside this region is not, and (2) the beam direction is always changed by $2\theta_B$.

(1) Reflection geometry. Firstly, we consider a Johansson-type curved-crystal monochromator (Fig. 5). The diffracting lattice plane makes an angle $\phi$ to the

Fig. 5. A Johansson-type curved crystal. (a) Geometry. S: X-ray source. F: Focusing point. $\Omega$: The whole angular width accepted by the curved crystal. $\omega_E$ and $\omega_F$: The angular widths of acceptance and emergence for diffraction at a tiny surface element. (b) The position–angle space representation. $A$: Window of acceptance. $E$: Window of emergence. $F$: X-ray beam at the focusing point.
surface. The spatial and angular coordinates are taken in a manner similar to the case of the flat perfect crystal. It is assumed that the X-ray source S, the crystal, and the focus F lie on the Rowland circle of radius R/2, where R is the radius of curvature of the diffracting lattice plane. Only one wavelength component \( \lambda_0 \) is considered here. The size of the crystal is assumed to be negligibly small compared with the source-to-crystal and crystal-to-focus distances \( p_0 \) and \( q_0 \), which are given by

\[
p_0 = R \sin (\theta_B + \phi),
\]

\[
q_0 = R \sin (\theta_B - \phi).
\]

If we assume that the angular widths of acceptance and emergence at any point on the crystal surface are infinitesimally small, we get \( x_h = x_0, \) \( x_0 = p_0 x_2 \) and \( x_4 = -q_0 x_4 \). However, in a real crystal finite angular widths \( \omega_B \) and \( \omega_F \) of acceptance and emergence should be taken into account. An X-ray beam of wavelength \( \lambda_0 \) is accepted in an angular range given by

\[
|\xi_0| \leq \omega_B/2
\]

and emitted in the angular range given by

\[
|\xi_h| \leq \omega_F/2,
\]

where

\[
\xi_0 = x_0/p_0.
\]

and

\[
\xi_h = -x_h/q_0.
\]

In general \( \omega_B \) and \( \omega_F \) are not equal to \( \omega_B \) and \( \omega_F \) of a flat perfect crystal, given by (8) and (9). But if the radius of curvature is large enough (roughly larger than a few meters), \( \omega_B \) and \( \omega_F \) in (18) and (19) can be replaced by \( \omega_B \) and \( \omega_F \) (Kaminaga, Matsushita & Kohra, 1978; Boeuf, Lagomarsino, Melone, Puliti & Rustichelli, 1978). Then a Johannsson-type curved-crystal monochromator is represented by window A of acceptance and E of emergence as shown in Fig. 5(b). \((x_0, x'_0)\) in A is transformed into \((x_h, x'_h)\) in E:

\[
\begin{pmatrix}
x_h \\
x'_h
\end{pmatrix} = \begin{pmatrix}
1/|b| & 0 \\
-1/f_c & |b|
\end{pmatrix} \begin{pmatrix}
x_0 \\
x'_0
\end{pmatrix}
\]

where

\[
1/f_c = 1/p_0 + 1/q_0.
\]

This transforming matrix does not coincide with \( T_c \), that of a thin lens in charged-particle beam optics, given by

\[
T_c = \begin{pmatrix}
1 & 0 \\
-1/f_c & 1
\end{pmatrix}
\]

where we take \( f > 0 \). Even when \(|b| = 1\), the matrix given in (22) cannot be regarded to accord with \( T_c \), since the sign of \( x_h \) is defined as opposite to that of \( x_0 \) for the curved crystal, while in the conventional charged-particle beam optics only one optical axis is considered and the sign of \( x_h \) should be the same as that of \( x_0 \). If we define \( x_0 \) and \( x_h \) to have the same signs with respect to optical axes SO and OF in Fig. 5(a), the transforming matrix for \(|b| = 1\) is given by

\[
T'_{ex} = \begin{pmatrix}
-1 & 0 \\
1/f_c & -1
\end{pmatrix}
\]

and this should be compared with \( T_L \). The area covered by A is equal to that covered by E. The whole angular aperture \( \Omega \) of the emergent beam is equal to that of the accepted beam if \( \Omega \gg \omega_B, \omega_F \). The X-ray beam at the focusing point is represented by \( F(F_1F_2F_3F_4) \) in Fig. 5(b). By taking into account the source-to-crystal and crystal-to-focus distances the X-ray source shown by \( (x_s, x'_s) \) in the x–x' space is transformed into \((x_f, x'_f)\) at the focus:

\[
\begin{pmatrix}
x_f \\
x'_f
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{pmatrix} \begin{pmatrix}
x_s \\
x'_s
\end{pmatrix}
\]

Equation (26b) shows that the magnifying or reducing property of the curved-crystal monochromator cannot be explained only by considering a single wavelength component \( \lambda_0 \). In order to explain the reducing property, the wavelength spread of the X-ray beam should be taken into account.

Johann- and logarithmic spiral-type crystals can be represented in the position–angle space if \( \xi'_0 \) and \( \xi'_h \) in (18) and (19) are replaced by \( \eta'_0 \) and \( \eta'_h \) for the Johann-type crystal and by \( \zeta'_0 \) and \( \zeta'_h \) for the logarithmic spiral one, where

\[
\eta'_0 = x_0/p_0 - \frac{1}{2}\left(\frac{x_0}{p_0}\right)^2 \csc (\theta_B + \phi),
\]

\[
\eta'_h = -x_h/q_0 - \frac{1}{2}\left(\frac{x_h}{q_0}\right)^2 \csc (\theta_B - \phi),
\]

\[
\zeta'_0 = x_0/p_0 - \frac{1}{2}|b| \left(\frac{x_0}{p_0}\right)^2 \sin 2\theta_B
\]

and

\[
\zeta'_h = -x_h|b|/p_0.
\]

Note that \( p_0 \) is not exactly given by (16) for a logarithmic spiral crystal. \( p_0 \) is the distance between the crystal and the X-ray source positioned on the caustic. Windows of acceptance and emergence of the Johann-type crystal are given by A and E in Fig. 6. The X-ray beam at the focus is represented by F. The focused beam has a finite spatial width since \( F \) is slightly curved. This is due to the geometrical aberration of the Johann-type crystal. A point \((x_0, x'_0)\) in the window of acceptance is transformed into a point \((x_h, x'_h)\) in the window of emergence, where \( x_h \) and \( x'_h \) are given by

\[
x_h = \frac{x_0 - q_0}{|b|} \left(\frac{x_0}{p_0}\right)^2 \left[\csc (\theta_B + \phi) + \csc (\theta_B - \phi)\right]
\]

and

\[
x'_h = |b| x'_0 - \frac{x_0}{f_c} + \frac{x_0^2}{2 p_0 f_c} \csc (\theta_B + \phi)
\]
\[ x_k = \frac{x_0}{|b|} - \frac{p_0}{2b^2} \left( \frac{x_0}{p_0} \right)^2 \sin 2\theta_B \]  
\[ x'_h = |b|x_0 - \frac{x_0}{f_z} + \frac{1}{2} \left( 1 + \frac{1}{|b|} \right) \left( \frac{x_0}{p_0} \right)^2 \sin 2\theta_B \]  
(33)

and

for the logarithmic spiral one.

(2) Transmission geometry. In order to show the basic optical property of the transmission-type focusing crystal, we will simplify the procedure by assuming that the diffracting lattice plane is normal to the surface and the aberration is neglected, and the radius of curvature is large. Then the windows of acceptance and of emergence in the position–angle space are both represented by

\[ \left| x' + \frac{x}{R' \cos \theta_B} \right| < \omega_{SL}/2 \]  
(35)

where \( R' (>0) \) is the radius of curvature of the cylindrically bent crystal and \( \omega_{SL} \) is the full width at half maximum of the symmetrical Laue-case diffraction curve. The transforming matrix from the window of acceptance to that of emergence is given by

\[ T_{ct} = \begin{pmatrix} -1 & 0 \\ 1/f_z & 1 \end{pmatrix} \]  
(36)

4.2 Elliptic total reflection mirror

An X-ray beam is reflected by a mirror as long as the glancing angle of the X-ray beam is smaller than the critical angle for total reflection at each point on the mirror surface. The deflected angle of the X-ray beam is not constant but depends upon the position on the mirror surface, in contrast with the case of curved crystals.

In order to get a representation of an elliptic total reflection mirror corresponding to that of a curved crystal, two optical axes \( SO \) and \( OF \) are considered in Fig. 7, where an X-ray source is located at \( S \), and \( F \) is the focusing point. The length of the mirror is assumed to be much shorter than the source-to-mirror and mirror-to-focus distances, \( p \) and \( q \), respectively. Hypothetical beam paths \( SP \) and \( QF \) are considered in place of the beam path \( SMF \). It can be shown that \( OP = OQ \) because \( \delta_\xi/\delta_\eta = q/p \), \( OP = p\delta_\xi \), and \( OQ = q\delta_\eta \), where \( \delta_\xi \) and \( \delta_\eta \) are angles between \( SO \) and \( SP \), and between \( OF \) and \( QF \), respectively. When the X-ray beam emitted from \( S' \) is incident on \( M \) making an angle \( x_0 \) to the optical axis \( SO \), the X-ray beam is reflected along \( MF' \) making an angle \( x_h \) to \( OF \). Then, the beam paths \( SP' \) and \( QF' \) should be considered. We can take \( OP' (=x_0) \) and \( OQ' (=x_h) \) as \( OP' = OP \) and \( OQ' = OQ \), which results in \( x_0 = x_h \), because we assumed that the mirror length is much smaller than \( p \) and \( q \). \( x_h \) is represented as a function of \( x_0 \) and \( x_h \), and the results can be summarized as

\[ \begin{pmatrix} x'h \\ x_h \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f_m & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0 \end{pmatrix} \]  
(37)

where

\[ 1/f_m = 1/p + 1/q \].

Although the expression given by (37) seems to be formally the same as that of a thin lens in charged-particle beam optics (Banford, 1966), we cannot regard the elliptic total reflection mirror as directly equivalent to a thin lens. We should remember that we took the sign of the \( Ox_h \) axis opposite to that of the \( Ox_0 \) axis. The transforming matrix is given by

\[ T_m = \begin{pmatrix} -1 & 0 \\ 1/f_m & -1 \end{pmatrix} \]  
(39)

Fig. 6. The position–angle space description of a Johann-type curved crystal. \( A, E \) and \( F \) have the same meanings as in Fig. 5(b).

Fig. 7. Geometry of an elliptic total reflection mirror. \( SS' \): X-ray source. \( FF' \): Focused image of \( SS' \). \( O\zeta_0 \): Optical axis for the incident beam. \( O\zeta_h \): Optical axis for the reflected beam.
if the signs of $Ox_0$ and $Ox_h$ axes are taken to be the same with respect to optical axes, as we discussed in § 3.1. This matrix corresponds to a combination of a flat mirror and a thin lens, since

$$
\begin{pmatrix}
-1 & 0 \\
1/f_m & -1
\end{pmatrix} = \begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
-1/f_m & 1
\end{pmatrix},
$$

(40)

where the first matrix on the right-hand side of (40) is that for a flat mirror and the second that for a lens.

In the case of a reducing mirror used in synchrotron-radiation optics, the length of the mirror will sometimes be comparable with $q$, so that $x_h$ cannot be regarded as equal to $x_0$. In such a case it would be necessary to take into account differences in $p$ and $q$ at different points on the mirror surface.

5. Discussion and conclusions

The approaches taken in the X-ray optical considerations can be compared with the geometrical theory of optical imaging of visible rays (Born & Wolf, 1970). In previous phase-space optical treatment of X-ray optical systems (Green, 1976; Pianetta & Lindau, 1976, 1977; Hastings, 1977), most expressions for optical elements were borrowed from the theory of charged-particle beam transport; where only one optical axis is considered, only the rays which lie in the immediate neighborhood of the optical axis are taken into account, focusing elements are treated as thin lenses, and terms involving squares or higher powers of off-axis displacements or the angle which rays make with the optical axis are neglected. This situation is analogous to the Gaussian optics of visible rays. On the other hand, the present approach corresponds to ray tracing. The position–angle correlations of X-ray beams accepted and emitted by optical elements are most accurately examined by tracing the X-ray beam paths. It should be noted that the adoption of two optical axes, one for the accepted beam and one for the emergent beam, made the treatment easier, for it became sufficient to treat only rays in the immediate neighborhood of the optical axes despite the fact that an X-ray beam is deflected by twice the Bragg angle by a crystal. Squares of off-axis displacements were taken into account for the Johann- and logarithmic spiral-type curved-crystal monochromators, so that geometrical aberrations were accurately depicted in representations of these crystals.

The following statement can be made in conclusion. The X-ray optical properties of flat and curved crystals are most accurately described in the position–angle space by taking into account, separately, angular widths of acceptance and emergence for diffraction. Focusing optical elements do not necessarily correspond directly to lenses in charged-particle beam optics. We have assumed that all the optical elements are thin. Thick optical elements can be treated as a set of thin optical elements placed at different distances in series. Ideas discussed in the present paper will be used as a basis for a consideration of X-ray optical systems for polychromatic radiation, using the position–angle–wavelength space in part II.

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