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Gauss–Schell Sources as Models for Synchrotron Radiation

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Gauss–Schell light sources are considered as models for synchrotron radiation. These sources can be viewed as random superpositions of coherent Gaussian beams. The relationships of the various widths that can be defined for the description of intensity and coherence, as related to the widths of the electron beam and the single-electron radiation (diffraction) pattern, are summarized. The description is also applied to the temporal coherence, which is of interest in the case of free-electron lasers.

Keywords: free-electron lasers; coherence; Gauss–Schell model.

1. Introduction

Gaussian beams (beams with a Gaussian transverse intensity distribution) are widely used as models for the propagation of monochromatic collimated coherent light beams (Arnaud, 1976). Gauss–Schell beams (or 'Gaussian Schell-model' beams, *i.e.* Gaussian beams with a Gaussian degree of coherence) are their natural generalization to partially coherent fields (Baltes & Steinle, 1977; Collett & Wolf, 1980). This is a very useful model as it may represent actual beams to a good approximation, and because it offers a qualitative insight into the physical properties of partially coherent beams by providing simple analytical relationships. Gori & Palma (1978) have shown that any Gauss–Schell source can be considered as an incoherent superposition of Gaussian coherent beams. We will call this the random superposition model.

In the case of synchrotron radiation, Kim (1986) has demonstrated that in four-dimensional transverse phase space the Wigner function of the synchrotron radiation emitted by an ensemble of randomly positioned electrons is the convolution of the Wigner function of the electron beam and of that of the intensity emitted by a single electron.

By approximating the spectral angular distribution of synchrotron radiation from a single electron with a Gaussian, synchrotron radiation can be described as a Gauss–Schell source, and this model can be useful in discussing the relationships among the various characteristic widths (of the electron beam, intensity, coherence *etc.*) that can be defined for the description of the angular and spatial coherence properties of synchrotron radiation. This description can be extended to the 'longitudinal' phase space (time-frequency), as this may be of interest in the case of free-electron lasers, where short electron bunches produce radiation with a coherence time close to the Fourier transform limit.

After a summary and discussion of the general concept of Gauss–Schell beams, we will outline the relationships among the various characteristic widths that can be defined for a bunch of synchrotron radiation in six-dimensional phase space, modelled as a Gauss–Schell source which is an incoherent convolution of the electron beam phase space and the single-electron phase-space distribution (diffraction pattern).

2. Mutual intensity and Wigner function

Let us summarize the definitions and symbols for the description of spatial or temporal coherence, where $p(x, y)$ are the transverse coordinates and z is the longitudinal coordinate. The spatial (transverse) coherence of $f(p)$ is described by the mutual intensity, Mf:

$$Mf(p, \Delta p) \equiv \langle f(p - \Delta p/2) f^*(p + \Delta p/2)\rangle, \qquad (1)$$

where $\langle \ldots \rangle$ is the ensemble average.

The degree of (spatial) coherence is defined as

$$\mu(p, \Delta p) \equiv Mf(p, \Delta p)/[I(p - \Delta p/2) I(p + \Delta p/2)]^{1/2}, \qquad (2)$$

where $I(p) = Mf(p, 0)$ is the field intensity.

From Fresnel's formula we obtain the propagation law for the mutual intensity,

$$Mf_z(p, \Delta p) = (\lambda z)^{-2} \int \mathrm{d}^2 p_0 \, \mathrm{d}^2 \Delta p_0 \, Mf_0(p_0, \Delta p_0) \times \exp[-ik(p - p_0)(\Delta p - \Delta p_0)/z], \qquad (3)$$

where f_z is the field in the transverse plane, z is a constant, and f_0 and p_0 are at $z = 0$.

In the *far* field (a sphere at infinity) the propagated field has a mutual intensity, expressed as a function of the variable $k_\perp = k\theta$ ($k_\perp$ is the transverse component of the wavevector $\mathbf{k}$ and θ is the angle between the direction of observation and the z axis) and of the difference $\Delta k_\perp$, which is related to the source ($z = 0$) mutual intensity by a Fourier transform:

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$$\begin{aligned}Mf_z(zk_\perp/k, z\Delta k_\perp/k) &= (\lambda z)^{-2} M\tilde{f}_0(k_\perp, \Delta k_\perp)\\ &= (\lambda z)^{-2} \int Mf_0(p, \Delta p)\\ &\quad \times \exp[ik_\perp.\Delta p + ip.\Delta k_\perp]\, d^2p\, d^2\Delta p.\end{aligned} \tag{4}$$

Note that in the Fourier transform, p is transformed into $\Delta k_\perp$ and Δp into $k_\perp$.

This choice of angular variable is that used in crystallography. With these variables (p and $k_\perp$ *etc.*) the volume in phase space is dimensionless, and the unity represents one mode of the electromagnetic field. The usual angular (two-dimensional) variable, θ, is obtained from

$$\theta = k_\perp/k = \lambda k_\perp/2\pi. \tag{5}$$

The Wigner function is the Fourier transform of the mutual intensity with respect to the variable Δp, *i.e.*

$$Wf(p, k_\perp) \equiv \int Mf(p, \Delta p) \exp(ik_\perp.\Delta p)\, d^2\Delta p. \tag{6}$$

It is a real (almost always positive) function and it can be considered as a phase-space energy density: note that this phase-space area is dimensionless. The Wigner function propagates in the same way as the radiance (or brightness) of geometrical optics, *i.e.*

$$Wf_z(p, k_\perp) = Wf_0(p - k_\perp z/k, k_\perp). \tag{7}$$

The Wigner function of a beam produced by an incoherent superposition of coherent beams with a phase-space distribution in position and angular divergence, $g(p, k_\perp)$, is the convolution of the Wigner function of the individual coherent beam, f_R, with the distribution, g (Gori & Palma, 1978; Kim, 1986),

$$Wf(p, k_\perp) = \int d^2\bar{p}_0\, d^2\bar{k}_\perp\, Wf_R(\bar{p}, \bar{k}_\perp) g(p - \bar{p}, k_\perp - \bar{k}_\perp), \tag{8}$$

while for the mutual intensity it is the convolution in p and the product in Δp,

$$Mf(p, \Delta p) = \int d^2\bar{p}\, Mf_R(\bar{p}, \Delta p)\tilde{g}(p - \bar{p}, \Delta p). \tag{9}$$

3. Gauss–Schell beams

A simple and general way of defining a Gauss–Schell beam (in n dimensions) is as a beam with a mutual intensity which is a ($2n$-dimensional) Gaussian, or, which is the same, its Wigner function is a Gaussian. For simplicity we express them in one dimension, x, while for the other dimensions, y and $t = z/c$, the expressions will be identical.† The consequences of the possible non-separability of the different variables will be qualitatively discussed later. This means, in general, that the mutual intensity, in two dimensions, for example x and $\Delta x = u$, can be written in the form

$$\begin{aligned}Mf(x, u) &= I(x)m(u)\exp(i\alpha xu)\\ &= N\exp(-x^2/2\sigma_I^2)\exp(-u^2/2\sigma_m^2)\exp(i\alpha xu),\end{aligned} \tag{10}$$

where I is the intensity, m is a function of u, N is a normalization constant and α is a parameter that provides a curvature of the wavefront.

For the whole four-dimensional or six-dimensional function we have for each variable (x, u, y, $v = \Delta y$, t, $\tau = \Delta t$) a corresponding width and for each couple of variables another parameter in the imaginary exponential. Each of these widths will be indicated by σ with indices indicating the quantity (I, m, μ). Other indices will be discussed later, e for the electron beam and R for single-electron radiation.

In the case where $\alpha = 0$ we say that the mutual intensity is *separable*, or satisfies Walther's condition (Walther, 1968), and this for x and u means that we are at the waist of the beam. In the waist the Wigner function is also separable, *i.e.*

$$Wf(x, \kappa) \propto I(x)J(\kappa) = N' \exp(-x^2/2\sigma_I^2)\exp(-\kappa^2/2\sigma_J^2), \tag{11}$$

where N' is a normalization constant and J is the intensity in the far field as a function of $\kappa = k_x$ and $\sigma_J = 1/\sigma_m$. The propagation of the Gauss–Schell beam from the waist at $z = 0$ is described by

$$\begin{aligned}Mf_z(x, u) &= N_z \exp[-x^2/2s_I^2(z)]\exp[-u^2/2s_m^2(z)]\\ &\quad \times \exp[-ikxu/R(z)],\end{aligned} \tag{12}$$

where $R = z(1 + k^2\sigma_I^2\sigma_m^2/z^2)$, $s_I(z) = \sigma_I(1 + z^2/k^2\sigma_m^2\sigma_I^2)^{1/2}$ and $s_m(z) = \sigma_m(1 + z^2/k^2\sigma_m^2\sigma_I^2)^{1/2}$, where $s_I(z)$ and $s_m(z)$ are the r.m.s. size of the intensity $I_z(x)$ and of the u-dependent term $m_z(u)$. The Rayleigh range is $\beta = k\sigma_I\sigma_m = k\sigma_I/\sigma_J$ and the emittance is $\sigma_I\sigma_J = \sigma_I/\sigma_m = s_I(z)/s_m(z)$.

Equation (12) shows that for all values of z during propagation from the source to the far field the ratio of mutual intensity width and intensity width remains constant (Friberg & Sudol, 1982).† This can also be seen as a consequence of Liouville's theorem (emittance = constant). In the far-field limit (and $\theta \ll 1$) this means

$$\begin{aligned}M\tilde{f}(\kappa, \Delta\kappa) &= J(\kappa)n(\Delta\kappa)\\ &= N'' \exp(-\kappa^2/2\sigma_J^2)\exp(-\Delta\kappa^2/2\sigma_n^2),\end{aligned} \tag{13}$$

where N'' is a normalization constant.

Then for a Gauss–Schell source we have the following reciprocity relations [a particular case of Walther (1968), Coïsson (1983) and Friberg & Wolf (1983)]:

$$\sigma_J = 1/\sigma_m, \qquad \sigma_n = 1/\sigma_I. \tag{14}$$

† This relationship is valid for perfectly collimated light beams only. We will discuss the effect of this in the final paragraph.

† We will see below [equation (16)] that the same holds for the ratio of the width of the degree of coherence to the intensity width.

It is easy to see that equation (10) in the waist ($\alpha = 0$) implies the Schell condition [$\mu(x, u)$ only function of u], and we can calculate the coherence width, σ_μ, in terms of the widths of mutual intensity,†

$$\begin{aligned} Mf(x, u) &\equiv [I(x + u/2)I(x - u/2)]^{1/2}\mu(u) \\ &= I_0 \exp\{-[(x + u/2)^2 + (x - u/2)^2]/4\sigma_I^2\}\mu(u) \\ &= I(x)\mu(u)\exp(-u^2/8\sigma_I^2) \\ &= I(x)m(u). \end{aligned} \quad (15)$$

The widths of $I(x)$, $\mu(u)$ and $m(u)$ are then related by

$$1/\sigma_m^2 = 1/\sigma_\mu^2 + 1/4\sigma_I^2. \quad (16)$$

In the far field an analogous relationship holds:

$$1/\sigma_n^2 = 1/\sigma_\nu^2 + 1/4\sigma_J^2. \quad (17)$$

When $\sigma_m \ll \sigma_I$ the beam is 'quasi-homogeneous' (Carter & Wolf, 1977; Goodman, 1985): in this case $\sigma_m \simeq \sigma_\mu$ and there is no need to distinguish between the two different 'coherence lengths'. Also in this case the far field is quasi-homogeneous [see (14) and (17)].

When the Gaussian size, σ_m, of the mutual intensity becomes comparable with the beam size, σ_I, the coherence factor is still Gaussian but with a coherence size different from σ_μ (16). In the limit $\sigma_m \to 2\sigma_I$ the beam is a completely coherent Gaussian beam ($\sigma_\mu \to \infty$).

In this section we have considered each couple of transverse variables separately, and then it is always possible to find a waist (the 'source'), while in four dimensions it is not possible in general. The separability of the transverse coordinates means that if we represent the $x - u$ or phase-space distribution with equal density curves, these are ellipses which have their principal axes along the coordinate axes.‡ In particular, the separability between two 'conjugate' variables (for example x with u for the mutual intensity or with κ for the Wigner function) means that the wavefront is flat (*i.e.* the beam is in the waist); if it is not, the angular distribution depends on position. If two pairs (x, u and y, v) are separable in two different positions of the optical axis then the beam is astigmatic. The non-separability between x, y (and Δx, Δy) means that the beam cross section is an ellipse with axes not parallel to the coordinate axes. Non-separability between x (u) and t (τ) means that the beam is not cross-spectrally pure (Mandel, 1961). Between t and ω the spectrum evolves in time during the pulse, or the pulse is 'chirped' (these two cases will be discussed below).

In conclusion, we have seen that the transverse coherence of a Gauss–Schell beam is characterized by six Gaussians: $I(x)$, $m(u)$ and $\mu(u)$, and those corresponding in the far field, $J(\kappa)$, $n(\Delta\kappa)$ and $\nu(\Delta\kappa)$, linked together by equations (16), (14) and (17).

† This equation shows that separability (Walther, 1968) and Gaussian intensity ↔ Gaussian intensity and Shell condition. It is easy to show that the Gaussian intensity is the only one that satisfies the Shell and Walther condition (Walther + Shell → Gaussian intensity):

$$[I(x + u)I(x - u)]^{1/2}\mu(2u) = Mf(x, 2u) = I(x)m(2u).$$

By applying the logarithm and by calling $h(u) = \log[\mu(2u)/m(2u)]$ and $L(x) = \log[I(x)]$, the previous equation becomes

$$2L(x) - L(x + u) - L(x - u) = 2h(u).$$

By Taylor expanding L around 0, then in order to have a dependence on u on the left-hand side the coefficients of the third or higher order must be 0 (*i.e.* constant, exponential or Gaussian intensity). In order to have a finite power light the second term must be smaller than 0, *i.e.* a Gaussian intensity.

‡ We will not consider twisted Gauss–Schell beams with phase $\exp[i(xv - yu)]$ that are not interesting for describing synchrotron radiation.

4. Random superposition model of electron beam and radiation

Let us now consider the synchrotron radiation beam as a random superposition of single-electron radiation pulses, making a random superposition model of our Gauss–Schell source. We can then find the relationships of the above-mentioned sizes with the characteristics of the electron beam (Coïsson, 1995).

Let the electron beam be described by a Gaussian distribution,† $g(x, \kappa)$, where σ_{Ie} and σ_{Je} are the electron-beam size and divergence and σ_{IR}, $\sigma_{JR} = 1/2\sigma_{IR}$ are the width and divergence of the coherent beams emitted by each electron (approximated to a Gaussian beam). The Gaussian approximation for the radiation from one electron is applicable when we restrict our attention to the central part of the spatial and angular distribution and, in the case of an undulator, for frequencies near the peaks (Kim, 1986).

From the intensity convolution theorem (Kim, 1986)

$$\sigma_I^2 = \sigma_{Ie}^2 + \sigma_{IR}^2, \qquad \sigma_J^2 = \sigma_{Je}^2 + \sigma_{JR}^2. \quad (18)$$

As this is a coherent beam ($\sigma_{\mu R} = \infty$), the spatial and angular intensity widths are related by diffraction (or the uncertainty principle), which in the Gaussian limit is‡

$$\sigma_{IR}\sigma_{JR} = 1/2. \quad (19)$$

In the t, ω phase space this is called the Fourier transform limit. The diffraction limit is reached when the electron beam is contained in a phase-space area smaller than this limit $\sigma_{Je} \ll \sigma_{JR}$ and $\sigma_{Ie} \ll \sigma_{IR} = 1/2\sigma_{JR}$.

5. Longitudinal phase space

Until now we have used a language adapted to the transverse dimensions: for the longitudinal dimensions (which can be labelled as z and $k_z \simeq k$ or t and ω) the relationships are the same, but the language may be a bit different and needs some comments about the physical interpretation.

If we write, as in the transverse case, the mutual intensity (self-coherence function) as $Mf(t, \tau) = Y(t)r(\tau)$, its double Fourier transform is $M\tilde{f}(\omega, \Omega) = S(\omega)C(\Omega)$, where $S(\omega)$ is the Fourier transform of $r(\tau)$ and $C(\Omega)$ of $Y(t)$.

We can define the pulse (intensity) length, σ_Y, the mutual intensity coherence time, σ_r, the coherence time,

$$\sigma_p = (1/\sigma_r^2 - 1/4\sigma_Y^2)^{-1/2}, \quad (20)$$

† Here it is assumed that the electron beam is in the waist but the model can be applied when the electron beam is converging or diverging. In this case the radiation beam will be converging or diverging.

‡ Remember that the intensity width is $1/2^{1/2}$ times the amplitude width.

the total (inhomogeneous) spectral (intensity) width,

$$\sigma_S = 1/\sigma_r, \tag{21}$$

and the two coherent (or 'homogeneous') spectral widths,

$$\sigma_C = 1/\sigma_Y, \tag{22}$$

$$\sigma_c = (1/\sigma_C - 1/4\sigma_S^2)^{-1/2}. \tag{23}$$

In the case of synchrotron radiation from bending magnets or undulators the coherence times are much shorter than the pulse length, so we are in the quasi-stationary case (intensity varies little over a coherence time), and the two coherence times σ_r and σ_ρ coincide, as well as the two homogeneous† spectral widths σ_C and σ_c. In the case of a free-electron laser we might be close to the Fourier transform limit, and the 'quasi-stationarity' condition does not apply.

Synchrotron radiation is a random superposition of wavetrains (which for our purposes we will approximate also as Gaussian, which is applicable when the undulator is apodized or we restrict our attention to the central part of the peak), each of duration σ_{YR}, and of spectral width σ_{SR}, for example for an undulator $\sigma_{YR} \simeq N\lambda/2^{1/2}c$ and $\sigma_{SR} \simeq c/2^{1/2}N\lambda$ [where N is a suitably defined number of periods (Coïsson, 1988)], emitted by an electron beam of duration σ_{Ye} and frequency spread $\sigma_{Se} = 2\Delta E/E$, where E is the electron energy and the factor of two comes from the E^2 dependence of the photon frequency.

The photon bunch duration is then

$$\sigma_Y \simeq (\sigma_{YR}^2 + \sigma_{Ye}^2)^{1/2}, \tag{24}$$

and the spectral width is

$$\sigma_S \simeq (\sigma_{SR}^2 + \sigma_{Se}^2)^{1/2}. \tag{25}$$

In the following we discuss the spatio-temporal correlations. Until now we have studied separately the field correlations in the transverse coordinates and in time. This is applicable when the transverse variables of the mutual intensity are separable from the temporal ones, *i.e.* the corresponding α parameters in (10) are zero. This means that representing the mutual intensity as a function of one spatial and one temporal variable, the ellipse has its axes along the coordinate axes, then the distributions of the two variables are not correlated. This kind of field is called *cross-spectrally pure* (Mandel, 1961). In other words we have treated the propagation of a quasi-monochromatic component of the spectrum. But when we consider the general (polychromatic) case, the propagation of each spectral component depends on wavelength: for the same coherence width, shorter wavelengths are more collimated than longer ones. This means that a source with a uniform spectrum beam will be more 'blue' at the centre after propagation (Dacic & Wolf, 1988). When the components at different wavelengths have beam sizes and coherence proportional to their wavelengths, the beam has the same spectral distribution (although in different scales) at any distance: in this case the beam is cross-spectrally pure. The radiation from an undulator is not cross-spectrally pure as the spectrum is angle-dependent.

Non-separability between t and ω means that the spectrum is time-varying (chirped): the longitudinal ellipse is oblique. It may be produced by an electron beam with an oblique longitudinal phase-space ellipse (a beam where the leading electrons are travelling at a speed different from that of the trailing ones) or by a special synchrotron radiation device producing a chirped single-electron radiation pulse, as for example a chirped (tapered) undulator (an undulator with a period varying along z).

† The word 'homogeneous' here has two different and conflicting meanings: in spectroscopy a 'homogeneous' spectrum is a coherent one (different spectral components having a correlated phase), while in the language of spatial coherence theory a 'quasi-homogeneous' spectrum would be an almost incoherent one.

6. Conclusions

In the case of synchrotron radiation we have an example of a source that in some cases can be described by a Gauss–Schell model in all three dimensions, the two transverse directions and time. For each Gauss–Schell distribution there is a corresponding far-field or reciprocal-space (or Fourier-transformed) Gauss–Schell distribution. Each Gauss–Schell distribution is characterized by three Gaussians [of which, due to equation (16), only two are independent]: the intensity, the Δx (or Δy or Δt)-dependent part of the mutual intensity and the degree of coherence. In total we have 18 Gaussians (12 for the spatial distributions and six for the temporal ones) but for equations (16) and (14) only six are independent.

We have also seen that these widths result from the combination of six Gaussian electron widths (beam size, divergence, bunch length *etc.*) and six single-electron radiation widths (8). In conclusion, we have 12 independent widths from which all the other 12 can be calculated.

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