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On the Σ classes in E^6

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In E^6, the cone of positive definite quadratic forms is subdivided into Σ_s subcones and its equivalence classes $\mathcal{E}_{\Sigma_s}$ are determined for s = 0–3, and 18–21.

1. Introduction

The discovery of quasicrystals, the structure of which can be viewed as projected from higher-dimensional translation lattices, has greatly stimulated the investigation of lattices and parallelohedra in arbitrary dimensions. The classification of the combinatorial types of primitive parallelohedra P induces a structure on the cone of positive definite quadratic forms $\mathcal{C}^+$.

In a series of papers, the shape of $\mathcal{C}$ and its subdivision into Φ and Σ subcones were discussed (Baburin & Engel, 2013; Engel, 2015, 2019).

Ryshkov (1973) defined the S subcone which contains all parallelohedra that have the same set of facet vectors $\mathcal{F}$, but without characterizing its boundary. The complete subcone was determined by Engel (2015) by the half-space intersection

$$\Sigma(\mathsf{P}) := \bigcap_{h=1}^{3N_b} \mathsf{H}_h. \quad (1)$$

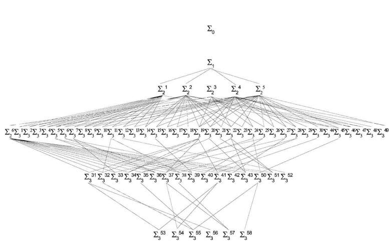
The investigation of translation lattices becomes most attractive in E^6 because many new phenomena appear for the first time in dimension 6.

In the report by Engel (2019), minimal and maximal Σ_s classes in E^6, $\mathcal{E}_{\Sigma_0}$ and $\mathcal{E}_{\Sigma_{21}}$, were investigated. The subscript 's' is an invariant of the class and denotes the number of closed zones of P [see Engel (2019), equations (12)–(13)]. This classification is continued for the Σ_s classes, s = 0, 1, 2, 3 and 18, 19, 20, 21.

The infinite family of Σ cones generate a face-to-face tiling of the cone $\mathcal{C}$ [see Engel (2019), equation (17) *ff.*]. In this tiling, for each class representative Σ_s^i are determined all the neighbouring Σ's adjacent to Σ_s^i by a common wall, in order to find new Σ_r^j. Proceeding in this way, for each class $\mathcal{E}_{\Sigma_s^k}$ can be found at least one representative Σ_s^k along a finite path of adjacent Σ's.

As a main result we obtain by this adjacency procedure:

For s = 0, 1, 2, 3 there exist 1, 1, 6, 58 Σ_s classes, and for s = 18, 19, 20, 21 there exist 15, 3, 1, 1 Σ_s classes in $\mathcal{C}$.



2. Determination of the Σ_s classes

Most concepts used in what follows were described by Engel (2019).

Beginning with Σ_0 as a representative of its class $\mathcal{E}_{\Sigma_0}$, and its subdivision into combinatorial Φ types, the Σ_s classes for s = 0, 1, 2, 3 are successively determined. Recall that Σ_0 has 216 walls $\mathsf{W}_i^{\Sigma_0}$, $i = 1, \cdots, 216$, which all are equivalent under the

Table 1
Wall normals n_i for the wall classes of Σ_1 in E^6.

Class	Order	$n_{11}, \ldots, n_{16}$ / $n_{22}, \ldots, n_{26}$ / $n_{33}, \ldots$ / n_{66}	Neighbour
1	1	0 −1 1 0 0 1 / 0 0 −1 1 −1 / 0 1 −1 1 / 0 0 1 / 0 −1 / −1	Σ_0
2	5	0 −1 1 −1 1 / 0 0 0 0 0 / 0 0 0 0 / 0 0 0 / 0 0 / 0	Σ_1
3	30	0 0 0 0 0 0 / 0 0 0 0 0 / 0 1 0 0 / 0 0 0 / 0 0 / 0	Σ_2^1
4	40	0 0 0 0 0 1 / 0 0 0 0 0 / 0 0 0 0 / 0 0 0 / 0 0 / 0	Σ_2^2
5	10	0 1 0 0 0 0 / 0 0 0 0 0 / 0 0 0 0 / 0 0 0 / 0 0 / 0	Σ_2^3
6	20	0 0 0 0 0 0 / 0 −1 0 0 0 / 1 0 0 1 / 0 0 0 / 0 0 / 0	Σ_2^4
7	60	0 0 0 0 0 0 / 0 0 0 0 0 / 0 0 0 0 / 0 0 0 / 0 1 / 0	Σ_2^5

group $\mathcal{G}_{E_6^*}$ (see Engel, 2019). The neighbouring Σ_s adjacent to Σ_0 are equivalent too, and were determined along the following steps:

Step S1: Within the class of equivalent walls, one wall $\mathsf{W}_l^{\Sigma_0}$, $1 \le l \le 216$, is selected, and for any $\Phi_j \subset \Sigma_0$ leaning on $\mathsf{W}_l^{\Sigma_0}$, the neighbouring Φ_k opposite to that wall is taken. Let $\mathsf{Q} \in \Phi_k^+$. The determination of $\Sigma_s(\mathsf{Q})$ requires first the computation of the primitive parallelohedron,

$$\mathsf{P}(\mathsf{Q}) := \bigcap_{\forall \mathbf{t} \in \Lambda^d \setminus \{O\}} \mathsf{H}_{\mathbf{t}}. \qquad (2)$$

Note that in E^6 every primitive parallelohedron has 126 facet vectors. The set of facet vectors of P is denoted by

$$\mathcal{F}_{\mathsf{P}} := \{\mathbf{f}_1, \cdots, \mathbf{f}_{126}\}. \qquad (3)$$

This shows that $\mathsf{P}(\mathsf{Q})$ has one closed zone with zone vector $\mathbf{z}^* = (0, 0, 0, 0, 0, 1)$, and thus it belongs to $\Sigma_1(\mathsf{Q})$. Because of symmetry, for each equivalent wall an equivalent result will be obtained.

Step S2: Next all triplets $\mathbf{f}_i, \mathbf{f}_j, \mathbf{f}_k \in \mathcal{F}_{\mathsf{P}}$ that fulfil the belt condition

$$\mathbf{f}_i + \mathbf{f}_j + \mathbf{f}_k = 0, \qquad (4)$$

are determined. Their number is $N_b = 371$, and thus,

$$\Sigma_1(\mathsf{Q}) := \bigcap_{h=1}^{3N_b} \mathsf{H}_h, \qquad (5)$$

is obtained. Because of the large number of half-spaces H_h, the direct calculation of Σ_1 is not practicable. Instead, the calculation of the Φ subcones inside Σ_1 will reveal the walls $\mathsf{W}_i^{\Sigma_1}$. Recall that Q is interior to Σ_1 if

$$\mathsf{Q} \in \mathsf{H}_h^+, \quad h = 1, \cdots, 3N_b. \qquad (6)$$

This allows the calculation of all $\Phi_k \subset \Sigma_1$ without explicitly knowing Σ_1, and for the walls of Σ_1 it holds that:

A wall $\mathsf{W}_j^{\Phi_k}$ of $\Phi_k \subset \Sigma_1$ is a wall of Σ_1 if there exists a wall H_h^0, $1 \le h \le 3N_b$, such that

$$\mathsf{W}_j^{\Phi_k} = \mathsf{H}_h^0. \qquad (7)$$

By calculating a sufficiently large number of $\Phi_k \subset \Sigma_1$, most of the walls of Σ_1 can be determined. The process converts relatively quickly.

Step S3: In order to verify the result, the *induced symmetry* of Σ_1 is applied:

For any $\mathsf{Q} \in \Sigma_1$ the induced symmetry of Σ_1 is defined by

$$\mathcal{G}_{\Sigma_1} := \left\{ \mathsf{S}_i \mid \mathsf{Q}_i = \mathsf{S}_i \mathsf{Q} \mathsf{S}_i^t \in \Sigma_1, \quad \forall \mathsf{S}_i \in \mathcal{G}_{E_6^*} \right\}. \qquad (8)$$

The centre

$$\mathsf{C} := \sum_{\forall \mathsf{Q}_i \in \Sigma_1} \mathsf{Q}_i, \qquad (9)$$

is invariant under the group $\mathcal{G}_{\Sigma_1}$ and lies in Σ_1. Applying the symmetry $\mathcal{G}_{\Sigma_1}$ to the walls of Σ_1 proves that there are 166 walls which belong to seven classes under $\mathcal{G}_{\Sigma_1}$, and these are shown in Table 1. The neighbouring Σ_s, $s = 0, 1, 2$, are given in Table 2.

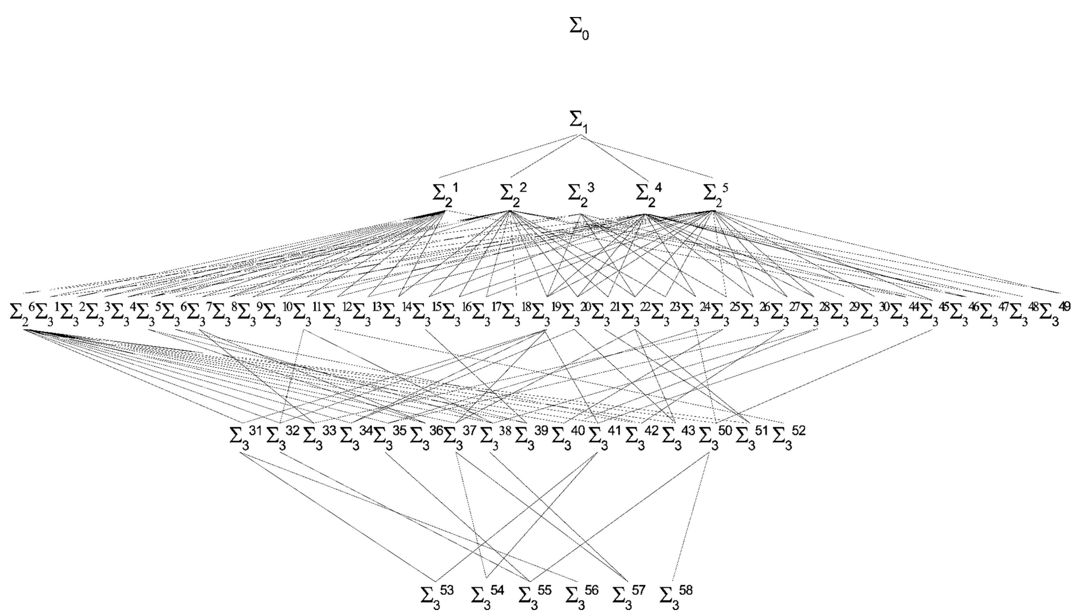


Figure 1
Shortest paths among the Σ_s classes for levels s = 0–3 in E^6.

Table 2
Σ classes for levels 0–3 in E^6.

s	Class	Subcone	Order	Neighbours
0	Σ_0	216	103860	$[216]\Sigma_1$
1	Σ_1	166	480	Σ_0, $[5]\Sigma_1$, $[30]\Sigma_2^1$, $[40]\Sigma_2^2$, $[10]\Sigma_2^3$, $[20]\Sigma_2^4$, $[60]\Sigma_2^5$
2	Σ_2^1	129	32	$[2]\Sigma_1$, $[4]\Sigma_2^1$, $[2]\Sigma_2^3$, $[8]\Sigma_3^1$, $[8]\Sigma_3^2$, $[8]\Sigma_3^3$, $[8]\Sigma_3^4$, $[8]\Sigma_3^5$, $[4]\Sigma_3^6$, $[16]\Sigma_3^7$, $[16]\Sigma_3^8$, $[8]\Sigma_3^9$, $[8+8]\Sigma_3^{10}$, $[4]\Sigma_3^{11}$, $[4]\Sigma_3^{12}$, $[8]\Sigma_3^{13}$, $[4]\Sigma_3^{14}$, Σ_3^{44}
	Σ_2^2	129	24	$[2]\Sigma_1$, $[2]\Sigma_2^2$, $[2]\Sigma_2^4$, $[6]\Sigma_2^5$, $[12]\Sigma_3^2$, $[3]\Sigma_3^3$, $[6]\Sigma_3^4$, $[6]\Sigma_3^5$, $[12]\Sigma_3^8$, $[6]\Sigma_3^{14}$, $[6]\Sigma_3^{15}$, $[6]\Sigma_3^{16}$, $[6]\Sigma_3^{17}$, $[6+6]\Sigma_3^{18}$, $[6]\Sigma_3^{19}$, $[6]\Sigma_3^{20}$, $[6]\Sigma_3^{21}$, $[6]\Sigma_3^{22}$, $[12]\Sigma_3^{23}$, $[3]\Sigma_3^{45}$, $[2]\Sigma_3^{46}$, Σ_3^{47}
	Σ_2^3	138	96	$[2]\Sigma_1$, $[6]\Sigma_2^1$, $[4]\Sigma_2^6$, $[24]\Sigma_3^5$, $[12]\Sigma_3^6$, $[24]\Sigma_3^{19}$, $[24]\Sigma_3^{22}$, $[12]\Sigma_3^{24}$, $[24]\Sigma_3^{25}$, $[6]\Sigma_3^{44}$
	Σ_2^4	134	48	$[2]\Sigma_1$, $[4]\Sigma_2^2$, $[6]\Sigma_2^5$, $[2]\Sigma_2^6$, $[24]\Sigma_3^7$, $[6]\Sigma_3^9$, $[12]\Sigma_3^{11}$, $[3]\Sigma_3^{12}$, $[6]\Sigma_3^{15}$, $[12]\Sigma_3^{19}$, $[12]\Sigma_3^{20}$, $[6]\Sigma_3^{21}$, $[3]\Sigma_3^{24}$, $[6]\Sigma_3^{26}$, $[6]\Sigma_3^{27}$, $[12]\Sigma_3^{28}$ $[6]\Sigma_3^{29}$, $[2]\Sigma_3^{46}$, Σ_3^{48}, $[3]\Sigma_3^{49}$
	Σ_2^5	125	16	$[2]\Sigma_1$, $[4]\Sigma_2^2$, $[2]\Sigma_2^4$, $[4]\Sigma_2^5$, $[8]\Sigma_3^1$, $[4]\Sigma_3^4$, $[2]\Sigma_3^6$, $[8]\Sigma_3^7$, $[8]\Sigma_3^8$, $[4]\Sigma_3^{10}$, $[8]\Sigma_3^{13}$, $[4+4]\Sigma_3^{16}$, $[8]\Sigma_3^{17}$, $[4]\Sigma_3^{20}$, $[4]\Sigma_3^{22}$, $[4]\Sigma_3^{23}$, $[8]\Sigma_3^{25}$, $[4]\Sigma_3^{26}$, Σ_3^{27}, $[8]\Sigma_3^{28}$, $[8+4]\Sigma_3^{30}$, $[4]\Sigma_3^{45}$, $[2]\Sigma_3^{49}$
	Σ_2^6	105.256280	12	Σ_2^3, Σ_2^4, Σ_2^6, $[3]\Sigma_3^{12}$, $[6]\Sigma_3^{31}$, $[6]\Sigma_3^{32}$, $[12]\Sigma_3^{33}$, $[6]\Sigma_3^{34}$, $[3]\Sigma_3^{35}$, $[12]\Sigma_3^{36}$, $[6]\Sigma_3^{37}$, $[6]\Sigma_3^{38}$, $[6]\Sigma_3^{39}$, $[12]\Sigma_3^{40}$, $[6]\Sigma_3^{41}$, $[3]\Sigma_3^{42}$, $[6]\Sigma_3^{43}$, $[3]\Sigma_3^{50}$, $[6]\Sigma_3^{51}$
3	Σ_3^1	129	4	Σ_2^1, $[2]\Sigma_2^5$, $[2]\Sigma_3^1$, Σ_3^2, $[2]\Sigma_3^4$, $[2]\Sigma_3^7$, $[2]\Sigma_3^8$, $[2]\Sigma_3^{22}$, $[115]\Sigma_4$
	Σ_3^2	103.497315	4	$[2]\Sigma_2^2$, Σ_3^1, $[2]\Sigma_3^2$, $[2]\Sigma_3^4$, $[2]\Sigma_3^7$, $[2]\Sigma_3^8$, $[2]\Sigma_3^{22}$, $[90]\Sigma_4$
	Σ_3^3	99.344740	8	$[2]\Sigma_2^1$, $[4]\Sigma_3^5$, $[2]\Sigma_3^9$, $[2]\Sigma_3^{12}$, $[89]\Sigma_4$
	Σ_3^4	100.361510	4	Σ_2^1, Σ_2^5, $[2]\Sigma_3^1$, $[2]\Sigma_3^2$, Σ_3^4, Σ_3^{11}, $[2]\Sigma_3^{13}$, Σ_3^{14}, $[1+1]\Sigma_3^{22}$, $[87]\Sigma_4$
	Σ_3^5	105.390562	4	Σ_2^1, Σ_2^3, $[1+1]\Sigma_3^3$, $[1+1]\Sigma_3^5$, Σ_3^6, Σ_3^9, Σ_3^{10}, Σ_3^{24}, $[2]\Sigma_3^{36}$, Σ_3^{39}, $[89]\Sigma_4$
	Σ_3^6	107.485226	8	Σ_2^1, Σ_2^3, $[2]\Sigma_3^5$, $[2+2]\Sigma_3^{10}$, Σ_3^{12}, Σ_3^{24}, $[2]\Sigma_3^{38}$, Σ_3^{42}, $[94]\Sigma_4$
	Σ_3^7	103.401973	2	Σ_2^1, Σ_2^4, Σ_2^5, Σ_3^1, Σ_3^2, $[1+1]\Sigma_3^7$, $[1+1+1]\Sigma_3^8$, Σ_3^{11}, Σ_3^{13}, Σ_3^{19}, Σ_3^{25}, Σ_3^{33}, Σ_3^{36}, $[87]\Sigma_4$
	Σ_3^8	98.307538	2	Σ_2^1, Σ_2^2, Σ_2^5, Σ_3^1, Σ_3^2, $[1+1+1]\Sigma_3^7$, $[1+1]\Sigma_3^8$, Σ_3^{13}, Σ_3^{14}, Σ_3^{19}, Σ_3^{25}, $[84]\Sigma_4$
	Σ_3^9	99.282678	8	$[2]\Sigma_2^1$, Σ_2^4, $[2]\Sigma_3^3$, $[2]\Sigma_3^5$, Σ_3^9, $[2]\Sigma_3^{10}$, Σ_3^{24}, $[88]\Sigma_4$
	Σ_3^{10}	99.311756	4	$[1+1]\Sigma_2^1$, Σ_2^5, Σ_3^3, Σ_3^5, $[1+1]\Sigma_3^6$, Σ_3^9, $[1+1]\Sigma_3^{10}$, Σ_3^{12}, $[88]\Sigma_4$
	Σ_3^{11}	109.471017	8	Σ_2^1, $[2]\Sigma_2^4$, $[2]\Sigma_3^4$, $[4]\Sigma_3^7$, $[2]\Sigma_3^{13}$, Σ_3^{14}, $[2]\Sigma_3^{19}$, $[2]\Sigma_3^{32}$, $[2]\Sigma_3^{38}$, $[91]\Sigma_4$
	Σ_3^{12}	112.627736	16	$[2]\Sigma_2^1$, Σ_2^4, $[2]\Sigma_2^6$, $[2]\Sigma_3^6$, $[4]\Sigma_3^{10}$, $[2]\Sigma_3^{12}$, $[2]\Sigma_3^{52}$, $[97]\Sigma_4$
	Σ_3^{13}	95.226124	4	Σ_2^1, $[2]\Sigma_2^5$, $[2]\Sigma_3^4$, $[2]\Sigma_3^7$, $[2]\Sigma_3^8$, Σ_3^{11}, Σ_3^{13}, Σ_3^{14}, $[2]\Sigma_3^{25}$, $[82]\Sigma_4$
	Σ_3^{14}	98.258158	8	Σ_2^1, $[2]\Sigma_2^2$, $[4]\Sigma_3^8$, $[2]\Sigma_3^4$, $[2]\Sigma_3^{19}$, $[2]\Sigma_3^{13}$, Σ_3^{11}, $[84]\Sigma_4$
	Σ_3^{15}	108.550026	8	$[2]\Sigma_2^2$, Σ_3^{15}, $[2]\Sigma_3^{16}$, $[2]\Sigma_3^{18}$, $[2]\Sigma_3^{20}$, $[4]\Sigma_3^{23}$, $[2]\Sigma_3^{26}$, $[2]\Sigma_3^{27}$, $[2]\Sigma_3^{39}$, $[89]\Sigma_4$
	Σ_3^{16}	94.214427	4	Σ_2^2, $[1+1]\Sigma_2^5$, Σ_3^{15}, $[1+2]\Sigma_3^{16}$, $[1+1]\Sigma_3^{18}$, $[1+1]\Sigma_3^{20}$, $[2]\Sigma_3^{23}$, Σ_3^{26}, Σ_3^{27}, $[2]\Sigma_3^{30}$, $[77]\Sigma_4$
	Σ_3^{17}	96.285460	4	Σ_2^2, $[2]\Sigma_2^5$, $[2]\Sigma_3^{17}$, $[2]\Sigma_3^{18}$, $[2]\Sigma_3^{21}$, $[2]\Sigma_3^{23}$, $[2+2]\Sigma_3^{28}$, $[2]\Sigma_3^{30}$, $[79]\Sigma_4$
	Σ_3^{18}	98.321970	4	$[1+1]\Sigma_2^2$, Σ_2^5, Σ_3^{15}, $[1+1]\Sigma_3^{16}$, $[2]\Sigma_3^{17}$, Σ_3^{18}, Σ_3^{20}, $[1+1[\Sigma_3^{21}$, $[2]\Sigma_3^{23}$, Σ_3^{26}, $[2]\Sigma_3^{30}$, $[81]\Sigma_4$
	Σ_3^{19}	112.567544	4	Σ_2^2, Σ_2^3, Σ_2^4, $[2]\Sigma_3^7$, $[2]\Sigma_3^8$, Σ_3^{11}, Σ_3^{14}, Σ_3^{19}, $[1+1]\Sigma_3^{22}$, $[2]\Sigma_3^{25}$, Σ_3^{31}, Σ_3^{32}, Σ_3^{34}, Σ_3^{37}, Σ_3^{41}, $[93]\Sigma_4$
	Σ_3^{20}	103.395491	4	Σ_2^2, Σ_2^4, Σ_2^5, Σ_3^{15}, Σ_3^{16}, $[1+1]\Sigma_3^{18}$, Σ_3^{20}, Σ_3^{21}, $[2]\Sigma_3^{23}$, Σ_3^{26}, $[2]\Sigma_3^{28}$, Σ_3^{29}, $[2]\Sigma_3^{30}$, Σ_3^{34}, Σ_3^{43}, $[84]\Sigma_4$
	Σ_3^{21}	103.387689	8	$[2]\Sigma_2^2$, $]\Sigma_2^4$, $[4]\Sigma_3^{17}$, $[2+2]\Sigma_3^{18}$, $[2]\Sigma_3^{20}$, $[2]\Sigma_3^{21}$, $[2]\Sigma_3^{28}$, Σ_3^{29}, $[2]\Sigma_3^{51}$, $[83]\Sigma_4$
	Σ_3^{22}	108.531296	4	Σ_2^2, Σ_2^3, Σ_2^5, $[2]\Sigma_3^1$, $[2]\Sigma_3^2$, $[1+1]\Sigma_3^4$, $[1+1]\Sigma_3^{19}$, Σ_3^{22}, $[2]\Sigma_3^{25}$, Σ_3^{37}, Σ_3^{43}, Σ_3^{51}, $[91]\Sigma_4$
	Σ_3^{23}	99.343145	4	$[2]\Sigma_2^2$, Σ_2^5, $[2]\Sigma_3^{15}$, $[2]\Sigma_3^{16}$, $[2]\Sigma_3^{17}$, $[2]\Sigma_3^{18}$, $[2]\Sigma_3^{20}$, $[2]\Sigma_3^{23}$, $[2]\Sigma_3^{28}$, Σ_3^{30}, $[81]\Sigma_4$
	Σ_3^{24}	114.572071	16	$[2]\Sigma_2^3$, Σ_2^4, $[2]\Sigma_3^6$, $[4]\Sigma_3^5$, $[2]\Sigma_3^9$, $[4]\Sigma_3^{31}$, $[2]\Sigma_3^{35}$, $[2]\Sigma_3^{50}$, $[95]\Sigma_4$
	Σ_3^{25}	104.367659	4	Σ_2^3, $[2]\Sigma_2^5$, $[2]\Sigma_3^7$, $[2]\Sigma_3^8$, $[2]\Sigma_3^{13}$, $[2]\Sigma_3^{19}$, $[2]\Sigma_3^{22}$, Σ_3^{25}, $[2]\Sigma_3^{33}$, $[2]\Sigma_3^{40}$, $[86]\Sigma_4$
	Σ_3^{26}	98.295360	8	Σ_2^4, $[2]\Sigma_2^5$, $[2]\Sigma_3^{15}$, $[2]\Sigma_3^{16}$, $[2]\Sigma_3^{18}$, $[2]\Sigma_3^{20}$, $[2]\Sigma_3^{27}$, Σ_3^{26}, $[4]\Sigma_3^{30}$, $[80]\Sigma_4$
	Σ_3^{27}	111.490894	16	$[2]\Sigma_2^4$, Σ_2^4, $[4]\Sigma_3^{15}$, $[4]\Sigma_3^{16}$, $[4]\Sigma_3^{26}$, $[2]\Sigma_3^{27}$, $[2]\Sigma_3^{35}$, $[2]\Sigma_3^{42}$, $[2]\Sigma_3^{52}$, $[88]\Sigma_4$
	Σ_3^{28}	99.298816	4	Σ_2^4, $[2]\Sigma_2^5$, $[2+2]\Sigma_3^{17}$, $[2]\Sigma_3^{20}$, Σ_3^{21}, $[2]\Sigma_3^{23}$, $[2+2]\Sigma_3^{28}$, Σ_3^{29}, $[2]\Sigma_3^{38}$, $[2]\Sigma_3^{40}$, $[78]\Sigma_4$
	Σ_3^{29}	113.641446	24	$[3]\Sigma_2^4$, $[6]\Sigma_3^{20}$, $[3]\Sigma_3^{21}$, $[6]\Sigma_3^{28}$, $[6]\Sigma_3^{41}$, $[89]\Sigma_4$
	Σ_3^{30}	90.184843	4	$[2+1]\Sigma_2^5$, $[2]\Sigma_3^{16}$, $[2]\Sigma_3^{17}$, $[2]\Sigma_3^{18}$, $[2]\Sigma_3^{20}$, Σ_3^{23}, $[2]\Sigma_3^{26}$, $[2]\Sigma_3^{28}$, $[2]\Sigma_3^{30}$, $[72]\Sigma_4$
	Σ_3^{31}	86.61862	4	Σ_2^6, Σ_3^{19}, Σ_3^{24}, $[2]\Sigma_3^{36}$, Σ_3^{37}, Σ_3^{38}, Σ_3^{53}, Σ_3^{56}, $[77]\Sigma_4$
	Σ_3^{32}	86.59029	4	Σ_2^6, $[2]\Sigma_3^{33}$, Σ_3^{37}, Σ_3^{11}, Σ_3^{56}, Σ_3^{57}, Σ_3^{19}, Σ_3^{38}, $[77]\Sigma_4$
	Σ_3^{33}	78.36807	2	Σ_2^6, Σ_3^7, Σ_3^{25}, Σ_3^{32}, Σ_3^{33}, Σ_3^{36}, Σ_3^{37}, $[71]\Sigma_4$
	Σ_3^{34}	84.57367	4	Σ_2^6, Σ_3^{19}, Σ_3^{20}, $[2]\Sigma_3^{40}$, Σ_3^{41}, $[1+1]\Sigma_3^{43}$, Σ_3^{51}, $[75]\Sigma_4$
	Σ_3^{35}	90.79939	8	Σ_2^6, Σ_3^{24}, Σ_3^{27}, Σ_3^{35}, $[2]\Sigma_3^{39}$, $[1+1]\Sigma_3^{42}$, Σ_3^{52}, $[81]\Sigma_4$
	Σ_3^{36}	78.36807	2	Σ_2^6, Σ_3^5, Σ_3^7, Σ_3^{31}, Σ_3^{33}, Σ_3^{36}, Σ_3^{38}, $[71]\Sigma_4$
	Σ_3^{37}	86.61862	4	Σ_2^6, Σ_3^{19}, Σ_3^{22}, Σ_3^{31}, Σ_3^{32}, $[2]\Sigma_3^{33}$, Σ_3^{54}, Σ_3^{57}, $[77]\Sigma_4$
	Σ_3^{38}	86.57513	4	Σ_2^6, Σ_3^{57}, Σ_3^{31}, $[2]\Sigma_3^{36}$, Σ_3^{11}, Σ_3^6, Σ_3^{32}, $[78]\Sigma_4$
	Σ_3^{39}	83.39029	4	Σ_2^6, Σ_3^5, Σ_3^{15}, Σ_3^{35}, $[1+1]\Sigma_3^{39}$, Σ_3^{42}, $[76]\Sigma_4$
	Σ_3^{40}	73.25013	2	Σ_2^6, Σ_3^{25}, Σ_3^{28}, Σ_3^{34}, $[1+1]\Sigma_3^{40}$, Σ_3^{41}, Σ_3^{43}, Σ_3^{51}, $[64]\Sigma_4$
	Σ_3^{41}	89.74566	4	Σ_2^6, Σ_3^{19}, Σ_3^{29}, Σ_3^{34}, $[2]\Sigma_3^{40}$, Σ_3^{41}, Σ_3^{43}, Σ_3^{51}, Σ_3^{53}, $[1+1]\Sigma_3^{54}$, $[77]\Sigma_4$
	Σ_3^{42}	89.73349	8	Σ_2^6, Σ_3^6, Σ_3^{27}, $[2]\Sigma_3^{34}$, $[1+1]\Sigma_3^{35}$, $[1+1]\Sigma_3^{52}$, $[80]\Sigma_4$
	Σ_3^{43}	84.57367	4	Σ_2^5, Σ_3^{20}, Σ_3^{22}, $[1+1]\Sigma_3^{34}$, $[2]\Sigma_3^{40}$, Σ_3^{41}, Σ_3^{51}, $[75]\Sigma_4$
	Σ_3^{44}	113.559629	32	Σ_2^1, $[2]\Sigma_2^3$, $[2]\Sigma_3^{44}$, $[4]\Sigma_3^{50}$, $[104]\Sigma_4$
	Σ_3^{45}	93.276584	8	Σ_2^2, $[2]\Sigma_2^5$, $[2+2]\Sigma_3^{45}$, $[2]\Sigma_3^{46}$, $[2]\Sigma_3^{47}$, $[2+2+4]\Sigma_3^{49}$, $[74]\Sigma_4$
	Σ_3^{46}	99.399178	24	$[2]\Sigma_2^2$, Σ_2^4, $[6]\Sigma_3^{45}$, $[1+2]\Sigma_3^{46}$, $[2]\Sigma_3^{47}$, Σ_3^{48}, $[3]\Sigma_3^{49}$, $[81]\Sigma_4$
	Σ_3^{47}	99.417350	72	$[3]\Sigma_2^2$, $[9]\Sigma_3^{45}$, $[6]\Sigma_3^{46}$, $[81]\Sigma_4$
	Σ_3^{48}	102.398634	144	$[3]\Sigma_2^4$, $[6]\Sigma_3^{46}$, $[9]\Sigma_3^{49}$, $[84]\Sigma_4$
	Σ_3^{49}	96.338198	16	Σ_2^4, $[2]\Sigma_2^5$, $[4+4]\Sigma_3^{45}$, $[2]\Sigma_3^{46}$, Σ_3^{48}, $[2+2]\Sigma_3^{49}$, $[78]\Sigma_4$
	Σ_3^{50}	89.80719	8	Σ_2^6, Σ_3^{24}, Σ_3^{44}, Σ_3^{50}, Σ_3^{55}, Σ_3^{58}, $[83]\Sigma_4$
	Σ_3^{51}	81.41501	4	Σ_2^6, Σ_3^{21}, Σ_3^{22}, Σ_3^{34}, $[2]\Sigma_3^{40}$, Σ_3^{41}, Σ_3^{43}, Σ_3^{51}, $[72]\Sigma_4$
	Σ_3^{52}	83.35433	8	Σ_3^{12}, Σ_3^{27}, Σ_3^{35}, $[1+1]\Sigma_3^{42}$, Σ_3^{52}, $[77]\Sigma_4$
	Σ_3^{53}	73.14453	8	$[2]\Sigma_3^{41}$, $[2]\Sigma_3^{31}$, $[2]\Sigma_3^{54}$, $[67]\Sigma_4$
	Σ_3^{54}	71.12163	4	Σ_3^{37}, $[1+1]\Sigma_3^{41}$, Σ_3^{53}, $[67]\Sigma_4$
	Σ_3^{55}	78.28445	16	$[2]\Sigma_3^{35}$, $[2]\Sigma_3^{50}$, $[2+2]\Sigma_3^{58}$, $[70]\Sigma_4$
	Σ_3^{56}	69.13979	8	$[2]\Sigma_3^{31}$, $[2]\Sigma_3^{32}$, $[2]\Sigma_3^{57}$, $[63]\Sigma_4$
	Σ_3^{57}	68.12463	4	Σ_3^{56}, Σ_3^{38}, Σ_3^{32}, Σ_3^{37}, $[64]\Sigma_4$
	Σ_3^{58}	75.21855	8	Σ_3^{35}, Σ_3^{42}, Σ_3^{50}, $[1+1]\Sigma_3^{55}$, $[70]\Sigma_4$

Table 3
Σ classes for levels 21–18 in E^6.

s	Class	Subcone	Types	Order	Neighbours
21	Σ_{21}	21.21	1	10080	$[21]\Sigma_{20}$
20	Σ_{20}	21.21	1	480	Σ_{21}, $[10]\Sigma_{19}^1$, $[10]\Sigma_{19}^2$
19	Σ_{19}^1	25.22	1/2	48	$[2]\Sigma_{20}$, $[8]\Sigma_{18}^1$, $[12]\Sigma_{18}^2$, $[3]\Sigma_{18}^3$
	Σ_{19}^2	21.21	1	48	Σ_{20}, Σ_{19}^2, Σ_{19}^3, $[4]\Sigma_{18}^1$, $[6]\Sigma_{18}^2$, $[4]\Sigma_{18}^4$, $[4]\Sigma_{18}^5$
	Σ_{19}^3	21.21	1	96	$[2]\Sigma_{19}^2$, Σ_{18}^6, $[6]\Sigma_{18}^7$, $[8]\Sigma_{18}^8$, $[4]\Sigma_{18}^9$
18	Σ_{18}^1	21.21	1	12	Σ_{19}^1, Σ_{19}^2, Σ_{18}^1, Σ_{18}^{10}, $[17]\Sigma_{17}$
	Σ_{18}^2	25.23	3/3	8	Σ_{19}^1, Σ_{19}^2, Σ_{18}^2, Σ_{18}^7, $[21]\Sigma_{17}$
	Σ_{18}^3	33.25	2/12	16	$[3]\Sigma_{19}^1$, $[30]\Sigma_{17}$
	Σ_{18}^4	25.22	2/2	24	$[2]\Sigma_{19}^2$, $[2]\Sigma_{18}^5$, $[2]\Sigma_{18}^8$, $[19]\Sigma_{17}$
	Σ_{18}^5	21.21	1	24	$[2]\Sigma_{19}^2$, $[2]\Sigma_{18}^4$, Σ_{18}^{11}, $[16]\Sigma_{17}$
	Σ_{18}^6	21.21	1	96	Σ_{19}^3, $[2]\Sigma_{18}^6$, $[18]\Sigma_{17}$
	Σ_{18}^7	27.24	4/5	16	Σ_{19}^3, $[2]\Sigma_{18}^2$, $[24]\Sigma_{17}$
	Σ_{18}^8	21.21	1	12	Σ_{19}^3, Σ_{18}^4, Σ_{18}^{12}, $[19]\Sigma_{17}$
	Σ_{18}^9	21.21	1	24	Σ_{19}^3, Σ_{18}^{13}, $[19]\Sigma_{17}$
	Σ_{18}^{10}	21.21	1	24	$[2]\Sigma_{18}^1$, Σ_{18}^{12}, $[18]\Sigma_{17}$
	Σ_{18}^{11}	21.21	1	24	Σ_{18}^5, $[2]\Sigma_{18}^{13}$, $[18]\Sigma_{17}$
	Σ_{18}^{12}	21.21	1	24	$[2]\Sigma_{18}^8$, Σ_{18}^{10}, $[18]\Sigma_{17}$
	Σ_{18}^{13}	21.21	1	12	Σ_{18}^9, Σ_{18}^{11}, Σ_{18}^{14}, $[18]\Sigma_{17}$
	Σ_{18}^{14}	21.21	1	24	$[2]\Sigma_{18}^{13}$, Σ_{18}^{15}, $[18]\Sigma_{17}$
	Σ_{18}^{15}	21.21	1	72	$[3]\Sigma_{18}^{14}$, $[18]\Sigma_{17}$

Step S4: For each Σ_s obtained, we proceed analogously to steps S1 to S3 in order to get further Σ_s. For each new Σ_s we have to check their equivalence:

Σ_s^k and Σ_s^l are arithmetically equivalent and belong to the same equivalence class $\mathcal{E}_{\Sigma_s^k}$ if there exists $\mathsf{A} \in GL_d(\mathbb{Z})$ such that for any $\mathsf{Q}_j \in \Sigma_s^l$ it holds that

$$\mathsf{Q}_i = \mathsf{A}\mathsf{Q}_j\mathsf{A}^t \in \Sigma_s^k. \quad (10)$$

Because $GL_d(\mathbb{Z})$ is of infinite order, the above equation is not practicable. However, if optimal bases are admitted to the forms Q only, then the number of transformations A that have to be taken into account becomes finite. It was discovered for every Σ_s at maximal path length 5 from Σ_0 (see Fig. 1) that it is sufficient to consider $\mathsf{A} \in \mathcal{G}_{E_6^*}$ only, in order to verify equivalence. If equivalence is proved for any Q_j then it holds for all $\mathsf{Q} \in \Sigma_s^l$.

Alternatively, the combinatorial equivalence of parallelohedra may be compared:

Σ_s^k and Σ_s^l are equivalent and belong to the same equivalence class if there exist $\mathsf{Q}_i \in \Sigma_s^k$ and $\mathsf{Q}_j \in \Sigma_s^l$ such that

$$\mathsf{P}(\mathsf{Q}_i) \overset{\text{comb}}{\simeq} \mathsf{P}(\mathsf{Q}_j). \quad (11)$$

The latter procedure requires a sufficiently large number of Φ cones to be determined in order to find at least one equivalent pair.

Analogously, using the procedures described in steps S1 to S4 the Σ_s cones, s = 21, 20, 19, 18, were successively determined starting with Σ_{21} as a representative of its class $\mathcal{E}_{\Sigma_{21}}$. Recall that Σ_{21} has 21 walls $\mathsf{W}_i^{\Sigma_{21}}$, $i = 1, \cdots, 21$, which all are equivalent under the group $\mathcal{G}_{A_6^*}$ (see Engel, 2019).

3. Results

In Table 2 are given the Σ_s classes, s = 0, 1, 2, 3, under the general linear group $GL_d(\mathbb{Z})$. Each equivalence class $\mathcal{E}_{\Sigma_s^i}$ is given by its representative Σ_s^i. $\Sigma_s^i(\mathsf{Q})$ is chosen such that Q becomes optimal. Under the heading 's' is given the number of closed zones. Under the heading 'Subcone' is stated the number of walls of Σ_s^i. In cases where the complete Σ_s cone was calculated, the numbers of walls and edges are indicated

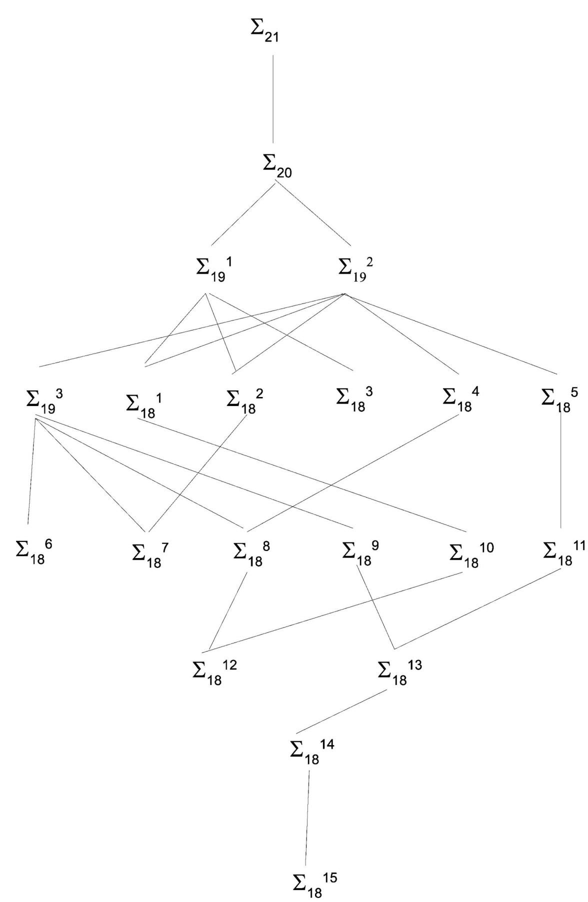


Figure 2
Shortest paths among the Σ_s classes for levels s = 21–18 in E^6.

as N_w and N_e, respectively. Under the heading 'Order' is given the order of the induced symmetry under $\mathcal{G}_{E_6^*}$. Under the heading 'Neighbours' are stated the neighbouring Σ_s^j, each of them preceded, in brackets, by the number of equivalent subcones under the group $\mathcal{G}_{\Sigma_s^i}$. If more than one number is given, it means that they are equivalent under $\mathcal{G}_{E_6^*}$. Remarkably, Σ_r^i has neighbours with $s = r - 1, r, r + 1$ only. Note that Σ_4 cones were not determined and the preceding number gives an upper bound for the number of equivalent subcones only. In Fig. 1 are drawn the shortest paths from Σ_0 to each other Σ_s^i.

In Table 3 are given the Σ_s classes, $s = 21, 20, 19, 18$. Under the heading 'Subcone' are given the numbers of walls N_w and edges N_e. Most of the cones are simple with Φ and Σ cones identical. In cases where the cone is not simple, two numbers are shown as '*a*/*b*' under the heading 'Types', where '*a*' indicates the number of Φ types and '*b*' gives the total number of Φ cones in Σ_s^i. The numbers of Φ types for $s = 21, 20, 19, 18$ correspond to the numbers given by Baburin & Engel (2013). All these Φ types correspond to principal primitive parallelohedra. Under the heading 'Order' is given the order of the induced symmetry under $\mathcal{G}_{A_6^*}$. Under the heading 'Neighbours' are listed the neighbouring Σ_s^j which are preceded, in brackets, by the number of equivalent types under the group $\mathcal{G}_{\Sigma_s^i}$. Note that the Σ_{17} cones were not determined and the preceding number gives an upper bound for the number of equivalent types only. In Fig. 2 are drawn the shortest paths from Σ_{21} to each other Σ_s^i.

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