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Lessons in Enumerative Combinatorics. By Ömer Eğecioğlu and Adriano M. Garsia. Springer, 2021. Softcover, pp. xvi + 479. ISBN 978-3-030-71252-5. Price EUR 68.56.

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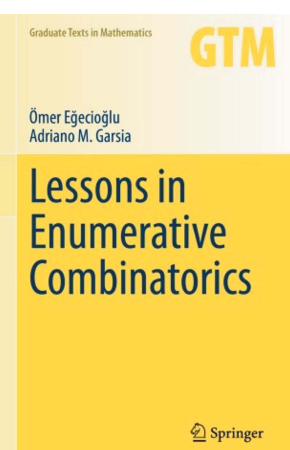
Enumerative combinatorics is the subject of study that deals with the number of ways certain patterns and arrangements can be formed. It is also the study of methods used for counting all the objects of a finite set, and that too without listing the actual objects one by one. It is a concern for all those who seek to know the answer to ‘how many’ without wanting to go through the painstaking process of counting one at a time.

Patterns and arrangements are the everyday concern of a crystallographer. Some of the most important crystallographic systems possess a considerably high degree of symmetry. Crystallography includes the problem of enumerating symmetry patterns that atoms in a crystal may form and that is when a crystallographer is required to work in the capacity of a combinatorist. The determination of coordination sequences and coronae, the orbit structure under inflation, or general complexity considerations are some of the combinatorial problems that sprout out of the study of crystals (Trebin, 2003). As such, it is no guess why a crystallographer should be asked to read a book on enumerative combinatorics. This book can sufficiently serve this purpose.

This book is an excellent exposition of both the basics and some intermediate-level concepts of enumerative combinatorics. The book has its foreword written by Richard Peter Stanley, the well known Emeritus Professor of Mathematics at the Massachusetts Institute of Technology whose two-volume magnum opus titled *Enumerative Combinatorics* is still considered to have no parallel. The preface to the book explains the several possible ways in which the book may be learnt or taught. The book has been developed into nine chapters, followed by a very brief section on reference books. This is followed by a bibliography containing references to 29 further readings. Each chapter ends with a large number of exercise problems (530 in all) and chapter-end quiz problems (45 in all).



Chapter 1 (74 pp.), titled *Basic Combinatorial Structures*, lays the foundation for the rest of the book. The chapter contains the rudimentary matter required for a deeper understanding of the principles that govern enumerative combinatorics. Anyone interested in mastering the art and science of counting must be familiar with the basic principles of counting detailed in the chapter. The author starts with the concept of a language and writes about the concepts of length, concatenation, lexicographic orders, set-theoretic notations and algebraic operations. Lattice paths are discussed in some detail. [*Note*: A lattice path is a path composed of connected horizontal and vertical line segments passing between adjacent grid points. It should not be confused with a Bravais lattice.] The concept of a lattice path is made interesting via the judicious use of revealing examples. The author then talks about Dyck languages and illustrates the usefulness of ballot sequences, nested parentheses and two-rowed Young tableaux. Injective and increasing words are discussed with suitable examples. Set partitions and restricted growth words are given due emphasis. Sterling numbers of the second kind, which count the number of ways of partitioning an integer into a specified number of blocks, are explained and exemplified. The chapter closes with illustrations of the usefulness of permutations and Stirling numbers of the first kind. The author has made sure that all the necessary requirements for a first encounter with combinatorics are met. This chapter is the longest chapter of the book, containing 121 exercise problems (almost 23 per cent of all the exercise problems) and 6 quiz problems. Chapter 2 (56 pp.), titled *Partitions and Generating Functions*, is about Ferrers diagrams (which represent partitions as patterns of dots), generating functions (a method of encoding infinite sequences of numbers by likening them to the coefficients of a formal power series), the Euler recursion for the partition function and the inversion-generating function for permutations. An in-depth discussion of parity and Gaussian polynomials is included. The chapter also includes a beautifully drafted section on miscellaneous identities wherein, through some gorgeous pictorial demonstrations, the author makes the readers understand the beauty of partitions.

Planar Trees and the Lagrange Inversion Formula constitutes the content of Chapter 3. This chapter (spanning 47 pp.) is a real treat to go through. Starting with an introduction to planar trees, the author talks about planar binary trees, tertiary trees and lattice-path representation of planar trees. The section on combinatorial enumeration of planar trees is very enriching and revealing. The chapter also includes a combinatorial proof of the Lagrange inversion formula, which the author introduces in the second chapter of the book. Arguably the most interesting and revealing chapter of the book is Chapter 4, titled *Cayley Trees*. After explaining the notions of monomial and the word of a Cayley tree, the author discusses the Prufer bijection. The enumeration of Cayley trees and Cayley forests is an excellent exposition of the powers of enumerative methods. The chapter also includes sections on functional digraphs and the Joyal encoding.

Chapter 5, titled *The Cayley Hamilton Theorem*, draws much of its content from linear algebra. It is the same length as the previous one. The chapter starts with a recollection of the Cayley–Hamilton theorem from linear algebra. For those that lack a fundamental grip on linear algebra, the chapter includes detailed proof of the theorem. The chapter also contains sections on the division method and the interpolation method, presented as an application of the preceding part of the chapter. *Exponential Structures and Polynomial Operators*, Chapter 6 with 52 pp., is a somewhat advanced chapter. It includes the discussion and exposition of further topics related to partitions, permutations, exponential structures and polynomial operators. The content of this chapter may be inaccessible to those who have not grasped the second chapter of the book well. It must not be attempted unless one feels comfortable with the subject matter of Chapter 2.

Chapter 7 (comprising 69 pp.) is titled *The Inclusion–Exclusion Principle*, a beautiful counting principle that makes the life of an enumerator a lot easier. Imagine one would like to enumerate all the numbers less than 1000 that are either multiple of 5 or of 6. Given that there are 200 multiples of 5 and 133 multiples of 6 that do not exceed 1000, it would not take one much to appreciate that numbers like 30, 60, 90 that are the multiples of 30 are counted twice and thus the double counting needs to be taken care of. This is precisely where the inclusion–exclusion principle comes in very handy. More complex situations related to enumeration are made accessible by the use of this principle. The chapter contains a detailed discussion of its general formulation and illustrates the principle with two classical examples – one on division properties and one on permutations without fixed points. This introduces the notion of derangements too. The chapter contains a dedicated section on the miscellaneous applications of the principle.

Chapter 8, the penultimate chapter, is *Graphs, Chromatic Polynomials and Acyclic Orientations*. Comprising 70 pp., it is devoted to the study of the rudiments of graph theory. It includes topics like graphs, chromatic polynomials, planar graphs and Platonic solids. Chapter 9 is about *Matching and Distinct Representations*. This chapter could be thought to be comprised of content that may be classified as recreational in nature.

In all, the book is an excellent source for anyone who wants to upgrade their knowledge of combinatorics. For crystallographers, the book may take their research to the next level, as a good working knowledge of how to count various structures and patterns could enhance their take on their research.

References

Trebin, H. R. (2003). Editor. *Quasicrystals: Structure and Physical Properties*. Weinheim: Wiley-VCH.