Similarity isometries of the diamond packing

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A lattice $\Gamma$ (of rank and dimension $d$) is a discrete subset of $\mathbb{R}^d$ that is the $\mathbb{Z}$-span of $d$ linearly independent vectors of $\mathbb{R}^d$ over $\mathbb{R}$. A (crystallographic) point packing generated by a lattice $\Gamma$ is a finite union of $\Gamma$ with distinct shifted copies of $\Gamma$ [1]. Point packings are models of crystals with multiple atoms per primitive unit cell. Examples of point packings include the honeycomb lattice, diamond lattice (crystal structure of diamond, tin, silicon, and germanium), and hexagonal closed packing (crystal structure of quartz). In particular, the diamond packing is the point packing formed by the union of the face-centered cubic (f.c.c.) lattice (white and gray dots), and its translate by a shift of $\frac{1}{4}(a,a,a)$ (black dots), where $a$ is the length of an edge of a conventional unit cell of the f.c.c. lattice (Fig. 1).

Figure 1. A conventional unit cell of the diamond packing.

A linear isometry $R$ of $\mathbb{R}^d$ is called a similarity isometry of a lattice $\Gamma$ if there exists a positive real number $\beta$ such that $\beta R \Gamma$ is a sublattice of finite index in $\Gamma$. The notion of a similarity isometry of a lattice was extended to point packings [2] as follows: A linear isometry $R$ of $\mathbb{R}^d$ is called a similarity isometry of a point packing $L$ if there exists a positive real number $\beta$ such that $\beta R L$ is a subset of $L$ with finite index in $L$, that is, the ratio of the density of points in $L$ by the density of points in $\beta R L$ is finite. Here, the set $\beta R L$ is called a similar subpacking of $L$, and we say that $\beta$ is a scaling factor of the similarity isometry $R$.

Using the algebra of quaternions, we determine the similarity isometries of the diamond packing and identify the similar subpackings generated and the possible scaling factors of each similarity isometry.


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