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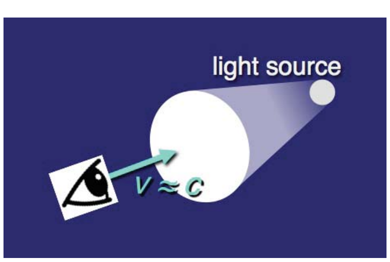

An enlightening procedure to explain the extreme power of synchrotron radiation

Giorgio Margaritondo*

Faculté des Sciences de Base, Ecole Polytechnique Fédérale de Lausanne (EPFL), CH 1015 Lausanne, Switzerland. *Correspondence e-mail: giorgio.margaritondo@epfl.ch

A simple approach exploits quantum properties to justify the dependence on γ^4 of the total synchrotron emitted power. It also clarifies some apparent puzzles and brings to light the underlying, multiple relativistic phenomena.

The relativistic arguments that explain the key properties of synchrotron radiation are sometimes tricky and potentially confusing for students (Margaritondo & Rafelski, 2017). One effective way to fight against the confusion is to use quantum properties (Margaritondo, 1995, 2019). We present here a specific example: a derivation of the total emitted power of a synchrotron source; and a comparison of the result with the Lorentz transformations for the energy of the radiation and its intensity.

The key point is of course that the total emitted power measured in the laboratory is proportional to γ^4 (Margaritondo, 1988; Mobilio *et al.*, 2015), where

$$\gamma = \frac{1}{\left[(1-(v^2/c^2)\right]^{1/2}} = \frac{E}{m_0c^2}, \tag{1}$$

v is the longitudinal speed of the emitting electron, E is the energy and m_0 is the rest mass. The γ^4-dependence is one of the most fundamental properties of synchrotron radiation (Margaritondo, 1988; Mobilio *et al.*, 2015), notably contributing to the extreme brightness and causing massive hadrons to emit much less than leptons.

However, the γ^4 factor may be somewhat puzzling: it is sometimes difficult for students to reconcile it with the Lorentz transformation of the wave energy from the electron reference frame to the laboratory frame (Rafelski, 2017; Steane, 2011; Einstein, 1905*b*):

$$\Omega = \left[\frac{1+(v/c)}{1-(v/c)}\right]^{1/2}\Omega' \simeq 2\gamma\Omega'. \tag{2}$$

Likewise, it might seem at odds with the transformation of the intensity (Einstein, 1905*b*)

$$I = \frac{1+(v/c)}{1-(v/c)}I' \simeq 4\gamma^2 I', \tag{3}$$

which, by the way, led Einstein as early as in 1905 to implicitly anticipate the high brightness of synchrotron radiation (Margaritondo & Rafelski, 2017; Einstein, 1905*a*).

Note that the total power emitted by an accelerated electron is a relativistic invariant. In fact, in the electron reference frame it is equal to the decrease per unit time of the electron energy m_0c^2. In another inertial frame the electron energy becomes $\gamma m_0 c^2$ and relativistic dilation changes the time also by a factor of γ, leaving the emitted power unchanged. This, however, does not help justifying its γ^4-dependence.

The standard way to derive the γ^4-dependence (Margaritondo, 1988) is based on the Lorentz transformations of time, coordinates and ultimately accelerations, and is quite straightforward. But it does not clarify the above apparent discrepancies. And, as we shall realize, it does not bring to light all the relativistic phenomena that underlie the γ^4-dependence.

Quantum properties offer an alternate approach, which besides being simple also helps us to solve the above puzzles. Consider a synchrotron radiation source and suppose that its emission is quantized (Einstein, 1905*a*), *i.e.* when detected it behaves like a collection of photons. Assume that a photon counter reveals the radiation that reaches its small capture area S, perpendicular to the (narrow) forward synchrotron beam (note that the transverse area S is Lorentz-invariant).

Suppose that the detector is synchronized with the electron bunches so that it counts only photons from electrons emitting at a distance D from S during a short time period δt. Furthermore, assume that the detector includes spectral filtering and only counts photons of energy $h\nu$, corresponding to the maximum of the emission spectrum. Defining $P(h\nu)$ as the emitted power at that photon energy measured in the laboratory reference frame, the photon counts during δt can be written as

$$N \propto \frac{P(h\nu)\,\delta t}{h\nu} \propto \frac{\delta t}{h\nu}. \tag{4}$$

Going now to the electron frame, the synchrotron emission is no longer confined to a narrow beam but occurs over a broad angular range – and the detector surface S only catches a portion. Assume, oversimplifying a bit, that the emission is isotropic: the portion that reaches S would be $S/(4\pi D'^2)$, where D' is the electron–detector distance in the electron frame. Compared with D, D' is Lorentz contracted (Rafelski, 2017; Steane, 2011),

$$D' = \frac{D}{\gamma}, \tag{5}$$

so the photon counts in the electron frame can be written as

$$N' \propto \frac{\delta t'}{h\nu'}\frac{S}{4\pi D'^2} = \frac{\delta t'}{h\nu'}\frac{S\gamma^2}{4\pi D^2}, \tag{6}$$

where $\delta t'$ and $h\nu'$ are the detection time and the detected photon energy measured in the electron frame. The more correct ‘cosine square’ (Larmor) law for the angular distribution of the emission in the electron reference frame would require calculation of a surface integral,[1] but would not change the essential point of equation (6),

$$N' \propto \frac{\delta t'}{h\nu'}\gamma^2. \tag{7}$$

[1] Specifically, the integral of $2\pi D'^2 \sin\theta$ (where θ is the angle with respect to the longitudinal direction), used to calculate the spherical surface area including S, has to be replaced by the integral of $2\pi D'^2 \sin\theta \cos^2\theta$, obtaining $4\pi D^2/3$ instead of $4\pi D^2$.

The photon counts given by equations (4) and (7) must of course coincide; after all, they could be read in a numerical display and the readings would be the same in both reference frames. Thus,

$$\frac{P(h\nu)\,\delta t}{h\nu} \propto \frac{\delta t'}{h\nu'}\gamma^2 \tag{8}$$

and

$$P(h\nu) \propto \left(\frac{\delta t'}{\delta t}\right)\left(\frac{\nu}{\nu'}\right)\gamma^2; \tag{9}$$

the two factors in parentheses in equation (9) are equal to γ (reciprocal time dilation) and, approximately, to 2γ (Doppler effect; Rafelski, 2017; Steane, 2011). Thus, we have

$$P(h\nu) \propto \gamma^4,$$

which also explains the γ^4-dependence of the total emitted power.

What do we learn from this alternate approach? First, that the γ^4-dependence of the emitted power cannot be derived from the transformation of the radiation energy alone, nor only from that of the intensity, because it is caused by a more complex combination of different relativistic phenomena, *i.e.* Lorentz contraction, time transformation and the Doppler effect.

By correctly treating these effects, one can realize that the above discrepancies are apparent, and in each case the treatment can be simplified by shortcuts offered by quantum properties.

For example, the radiation energy corresponding to photons of energy $h\nu$ can be written as a photon count multiplied by $h\nu$. Thus, since photon counts are invariant, the radiation energy and the frequency must transform in the same way. This explains why the factor of equation (2) is the same as that of the Doppler effect (Rafelski, 2017).

Regarding the intensity, note that it is equal to the radiation energy density multiplied by the invariant c, so it must transform like the energy density. Consider now a volume defined by S and by a length equal to n wavelengths $\lambda = c/\nu$ (in the laboratory frame), which contains A photons of energy $h\nu$. The energy density corresponding to such photons is $Ah\nu/Sn(c/\nu) = [Ah/(Snc)]\nu^2$ in the laboratory frame and $[Ah/(Snc)]\nu'^2$ in the electron frame. Thus, since the term Ah/Snc only includes invariants, the energy density and therefore the intensity must transform in the same way as ν^2 – justifying equation (3), whose factor is indeed the square of the Doppler factor.

Finally, to correctly relate the intensity to the total power one must not make the mistake of assuming that they transform in the same way because S is invariant. In fact, as explained above, the solid angle captured by S is *not* invariant due to the Lorentz contraction. And this, as we have seen, introduces the additional factor proportional to γ^2 that transforms equation (3) into the γ^4-dependence of the emitted power.

This is, in general terms, a specific example of a more systematic aspect (Margaritondo & Rafelski, 2017; Margaritondo, 1988); many important properties of synchrotron radiation are the result of not one but several combined relativistic effects. Only by taking into account all these effects can one avoid apparent contradictions and serious mistakes.

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References

Einstein, A. (1905*a*). *Ann. Phys.* **322**, 132–148.
Einstein, A. (1905*b*). *Ann. Phys.* **322**, 891–921.
Margaritondo, G. (1988). *Introduction to Synchrotron Radiation*. New York: Oxford University Press.
Margaritondo, G. (1995). *Eur. J. Phys.* **16**, 169–171.
Margaritondo, G. (2019). *Eur. J. Phys.* **40**, 035402.
Margaritondo, G. & Rafelski, J. (2017). *J. Synchrotron Rad.* **24**, 898–901.
Mobilio, S., Boscherini, F. & Meneghini, C. (2015). *Synchrotron Radiation Basics, Methods and Applications*. Berlin: Springer.
Rafelski, J. (2017). *Relativity Matters: from Einstein's EMC2 to Laser Particle Acceleration and Quark-Gluon Plasma*. Berlin: Springer.
Steane, A. (2011). *Relativity made relatively easy*, https://users.physics.ox.ac.uk/~Steane/teaching/rel_intro.pdf.